Online Appendix for Multiproduct-Firm Oligopoly: An Aggregative Games Approach

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Contents

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I Quasi-Linear Integrability

I.1 Statement of the Theorem

Our goal is to prove the following result:

Theorem I. *Let $\mathcal{N}$ be a finite and non-empty set. For every $k \in \mathcal{N}$, let h_k (resp. g_k) be a $\mathcal{C}^2$ (resp. $\mathcal{C}^1$) function from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$. Suppose that $h'_k < 0$ for every k. Define demand system D as follows:*

$$D_k\left((p_j)_{j\in\mathcal{N}}\right) = \frac{g_k(p_k)}{\sum_{j\in\mathcal{N}} h_j(p_j)}, \quad \forall k \in \mathcal{N}, \quad \forall \left(p_j\right)_{j\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$$

The following assertions are equivalent:

(i) D is quasi-linearly integrable.

(ii) There exists a strictly positive scalar α such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h'_k$. Moreover, $h''_k > 0$ for every $k \in \mathcal{N}$, and $\sum_{k\in\mathcal{N}} \gamma_k \leq \sum_{k\in\mathcal{N}} h_k$.

When this is the case, function $v(.)$ is an indirect subutility function for the associated demand system if and only if there exists $\beta \in \mathbb{R}$ such that $v(p) = \alpha \log\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right) + \beta$ for every $p >> 0$.

I.2 Preliminary Technical Lemmas

We first state and prove two preliminary technical lemmas, which will be useful to prove Theorem I:

Lemma I. *For every $n \geq 1$, for every $(\alpha_i)_{1\leq i\leq n} \in \mathbb{R}^n$, define*

$$\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right) = \begin{pmatrix} 1-\alpha_1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{pmatrix}$$

Then,[1]

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\leq i\leq n}\right)\right) = (-1)^n \left(\left(\prod_{k=1}^{n} \alpha_k\right) - \sum_{j=1}^{n} \left(\prod_{\substack{1\leq k\leq n \\ k\neq j}} \alpha_k\right)\right)$$

[1] We adopt the convention that the product of an empty collection of real numbers is equal to 1.

Moreover, matrix $\mathcal{M}\left((\alpha_i)_{1\le i\le n}\right)$ *is negative semi-definite if and only if* $\alpha_i \geq 1$ *for all* $1 \leq i \leq n$ *and*

$$\sum_{i=1}^{n} \frac{1}{\alpha_i} \leq 1.$$

Proof. We prove the first part of the lemma by induction on $n \geq 1$. Start with $n = 1$. Then,

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\le i\le n}\right)\right) = 1 - \alpha_1 = (-1)^1(\alpha_1 - 1),$$

so the property is true for $n = 1$.

Next, let $n \geq 2$, and assume the property holds for all $1 \leq m < n$. By n-linearity of the determinant,

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\le i\le n}\right)\right) = (-\alpha_1)\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1-\alpha_n \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix}.$$

Applying Laplace's formula to the first column, we can see that the first determinant is, in fact, equal to $\det\left(\mathcal{M}\left((\alpha_i)_{2\le i\le n}\right)\right)$. The second determinant can be simplified by using n-linearity one more time:

$$\begin{aligned} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1-\alpha_2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix} &= -\alpha_2\begin{vmatrix} 1 & 0 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1-\alpha_n \end{vmatrix} + \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n \end{vmatrix}, \\ &= -\alpha_2 \det\left(\mathcal{M}\left(0, (\alpha_i)_{3\le i\le n}\right)\right) + 0, \end{aligned}$$

where the second line follows again from Laplace's formula and from the fact that the first two rows of the second matrix in the first line's right-hand side are colinear. Therefore,

$$\begin{aligned} \det\mathcal{M}\left((\alpha_i)_{1\le i\le n}\right) =& -\alpha_1 \det\left(\mathcal{M}\left((\alpha_i)_{2\le i\le n}\right)\right) - \alpha_2 \det\left(\mathcal{M}\left(0, (\alpha_i)_{3\le i\le n}\right)\right), \\ =& -\alpha_1(-1)^{n-1}\left(\left(\prod_{k=2}^{n}\alpha_k\right) - \sum_{j=2}^{n}\left(\prod_{\substack{2\le k\le n \\ k\neq j}}\alpha_k\right)\right) \\ & -\alpha_2(-1)^{n-1}\left(0 - \prod_{k=3}^{n}\alpha_k\right), \end{aligned}$$

$$= (-1)^n \left(\left(\prod_{k=1}^{n} \alpha_k \right) - \sum_{j=2}^{n} \left(\prod_{\substack{1 \le k \le n \\ k \neq j}} \alpha_k \right) - \prod_{k=2}^{n} \alpha_k \right),$$

$$= (-1)^n \left(\left(\prod_{k=1}^{n} \alpha_k \right) - \sum_{j=1}^{n} \left(\prod_{\substack{1 \le k \le n \\ k \neq j}} \alpha_k \right) \right).$$

We now turn our attention to the second part of the lemma. Assume first that matrix $\mathcal{M}\left((\alpha_i)_{1\le i\le n}\right)$ is negative semi-definite. Then, all its diagonal terms have to be non-positive, i.e., $\alpha_i \geq 1$ for all i. Besides, the determinant of this matrix should be non-negative (resp. non-positive) if n is even (resp. odd). Put differently, the sign of the determinant should be $(-1)^n$ or 0. Since the α's are all different from zero, this determinant can be simplified as follows:

$$\det\left(\mathcal{M}\left((\alpha_i)_{1\le i\le n}\right)\right) = (-1)^n \left(\prod_{k=1}^{n} \alpha_k \right) \left(1 - \sum_{k=1}^{n} \frac{1}{\alpha_k} \right).$$

This expression has sign $(-1)^n$ or 0 if and only if $\sum_{k=1}^{n} \frac{1}{\alpha_k} \leq 1$.

Conversely, assume that the α's are all ≥ 1, and that $\sum_{k=1}^{n} \frac{1}{\alpha_k} \leq 1$. The characteristic polynomial of matrix $\mathcal{M}\left((\alpha_i)_{1\le i\le n}\right)$ is defined as

$$P(X) = \begin{vmatrix} 1-\alpha_1-X & 1 & \cdots & 1 \\ 1 & 1-\alpha_2-X & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1-\alpha_n-X \end{vmatrix}.$$

This determinant can be calculated using the first part of the lemma. For every $X > 0$,

$$\begin{aligned} (-1)^n P(X) &= \underbrace{\left(\prod_{k=1}^{n} (\alpha_k + X) \right)}_{>0} \left(1 - \sum_{k=1}^{n} \frac{1}{\alpha_k + X} \right), \\ &> \underbrace{\left(\prod_{k=1}^{n} (\alpha_k + X) \right)}_{>0} \underbrace{\left(1 - \sum_{k=1}^{n} \frac{1}{\alpha_k} \right)}_{\geq 0}, \\ &> 0. \end{aligned}$$

Therefore, $P(X)$ has no strictly positive root, matrix $\mathcal{M}\left((\alpha_i)_{1\le i\le n}\right)$ has no strictly positive eigenvalue, and this matrix is therefore negative semi-definite. □

Lemma II. *Let M be a symmetric n-by-n matrix, $\lambda \neq 0$, and $1 \leq k \leq n$. Let A^k be the matrix obtained by dividing the k-th line and the k-th column of M by λ. Then, M is negative* *semi-definite if and only if A^k is negative semi-definite.*

Proof. Suppose M is negative semi-definite, and let $X \in \mathbb{R}^n$. Write A^k as $(a_{ij})_{1\leq i,j\leq n}$ and M as $(m_{ij})_{1\leq i,j\leq n}$. Finally, define Y as the n-dimensional vector obtained by dividing X's k-th component by λ. Then,

$$\begin{aligned}
X'A^kX &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j, \\
&= \left(\sum_{\substack{1\leq i\leq n \\ i\neq k}}\sum_{\substack{1\leq j\leq n \\ j\neq k}} a_{ij}x_ix_j\right) + 2x_k \sum_{\substack{1\leq i\leq n \\ i\neq k}} a_{ik}x_i + x_k^2 a_{kk}, \\
&= \left(\sum_{\substack{1\leq i\leq n \\ i\neq k}}\sum_{\substack{1\leq j\leq n \\ j\neq k}} m_{ij}x_ix_j\right) + 2\frac{x_k}{\lambda} \sum_{\substack{1\leq i\leq n \\ i\neq k}} m_{ik}x_i + \left(\frac{x_k}{\lambda}\right)^2 m_{kk}, \\
&= \left(\sum_{\substack{1\leq i\leq n \\ i\neq k}}\sum_{\substack{1\leq j\leq n \\ j\neq k}} m_{ij}y_iy_j\right) + 2y_k \sum_{\substack{1\leq i\leq n \\ i\neq k}} m_{ik}y_i + y^2 m_{kk}, \\
&= Y'MY, \\
&\leq 0, \text{ since } M \text{ is negative semi-definite.}
\end{aligned}$$

Therefore, A^k is negative semi-definite.

The other direction is now immediate, since M can be obtained by dividing the k-th line and the k-th column of matrix A^k by $1/\lambda$. □

I.3 Proof of Theorem I

Proof. To simplify notation, assume without loss of generality that $\mathcal{N} = \{1, \ldots, n\}$, and let $D(.)$ be the demand system associated with the demand component under consideration. For every $p >> 0$, put $J(p) = \left(\frac{\partial D_i}{\partial p_j}(p)\right)_{1\leq i,j\leq n}$. Theorem 1 in Nocke and Schutz (2016) states that D is quasi-linearly integrable if and only if $J(p)$ is symmetric and negative semi-definite for every $p >> 0$.

We first show that matrix $J(p)$ is symmetric for every p if and only if there exists a strictly positive scalar α such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h_k'$. If $J(p)$ is symmetric for every p, then, for every $1 \leq i, j \leq n$ such that $i \neq j$, for every $p >> 0$,

$$-\frac{h_j'(p_j)g_i(p_i)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2} = J_{i,j}(p) = J_{j,i}(p) = -\frac{h_i'(p_i)g_j(p_j)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2}.$$

It follows that, for every $1 \leq i \leq n$, for every $x > 0$,

$$\frac{h_i'(x)}{g_i(x)} = \frac{h_1'(1)}{g_1(1)} \equiv -\beta \tag{i}$$

If $\beta = 0$, then $h_i' = 0$ for every i, which violates the assumption that h_i is strictly decreasing. Therefore, $\beta \neq 0$, and we can define $\alpha \equiv 1/\beta$. It follows that $g_i = -\alpha h_i'$. Since $g_i > 0$ and $h_i' \leq 0$, we can conclude that $\alpha > 0$. Conversely, if there exists a strictly positive scalar α such that, for every $k \in \mathcal{N}$, $g_k = -\alpha h_k'$, then, for every $1 \leq i, j \leq n$, $i \neq j$, for every $p >> 0$,

$$J_{i,j}(p) = -\frac{h_j'(p_j) g_i(p_i)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2} = \alpha \frac{h_j'(p_j) h_i'(p_i)}{\left(\sum_{k\in\mathcal{N}} h_k(p_k)\right)^2} = J_{j,i}(p),$$

and matrix $J(p)$ is therefore symmetric for every p.

Next, suppose that there exists $\alpha > 0$ such that, for every $1 \leq k \leq n$, $g_k = -\alpha h_k'$. We want to show that $J(p)$ is negative semi-definite for every $p >> 0$ if and only if $h_k'' > 0$ for every $1 \leq k \leq n$, and $\sum_{k=1}^n \gamma_k \leq \sum_{k=1}^n h_k$.

Fix $p >> 0$. To ease notation, we write $h_k = h_k(p_k)$ for every k, and define $H \equiv \sum_{k\in\mathcal{N}} h_k$. We obtain the following expression for matrix $J(p)$:

$$J(p) = \frac{\alpha}{H^2} \begin{pmatrix} (h_1')^2 - h_1'' H & h_1' h_2' & \cdots & h_1' h_n' \\ h_2' h_1' & (h_2')^2 - h_2'' H & \cdots & h_2' h_n' \\ \vdots & \vdots & \ddots & \vdots \\ h_n' h_1' & h_n' h_2' & \cdots & (h_n')^2 - h_n'' H \end{pmatrix}.$$

$J(p)$ is negative semi-definite if and only if

$$\begin{pmatrix} (h_1')^2 - h_1'' H & h_1' h_2' & \cdots & h_1' h_n' \\ h_2' h_1' & (h_2')^2 - h_2'' H & \cdots & h_2' h_n' \\ \vdots & \vdots & \ddots & \vdots \\ h_n' h_1' & h_n' h_2' & \cdots & (h_n')^2 - h_n'' H \end{pmatrix}$$

is negative semi-definite. Applying Lemma II n times (by dividing row k and column k by h_k', $1 \leq k \leq n$), this is equivalent to matrix

$$\begin{pmatrix} 1 - \frac{h_1''}{(h_1')^2} H & 1 & \cdots & 1 \\ 1 & 1 - \frac{h_2''}{(h_2')^2} H & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 - \frac{h_n''}{(h_n')^2} H \end{pmatrix}$$

being negative semi-definite. By Lemma I, this holds if and only if $\frac{h_k''}{\left(h_k'\right)^2}H \geq 1$ for all $1 \leq k \leq n$, and $\frac{1}{H}\sum_{k=1}^n \frac{\left(h_k'\right)^2}{h_k''} \leq 1$. This is equivalent to $h_k'' > 0$ for all k, and $\sum_{k=1}^n \gamma_k \leq \sum_{k=1}^n h_k$.

Finally, Nocke and Schutz (2016) show that, v is an indirect subutility function for demand system D if and only if $\nabla v = -D$. Clearly, this is equivalent to

$$v(p) = \alpha \log\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right) + \beta, \quad \forall p >> 0,$$

where $\beta \in \mathbb{R}$ is a constant of integration. □

Proposition 1 is then an immediate corollary of Theorem I.

II Pricing Game: Preliminaries

II.1 Preliminary Technical Lemma

Let $\mathcal{H}$ be the set of $\mathcal{C}^3$, strictly decreasing and log-convex functions from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$. Let $\mathcal{H}^\iota$ be the set of functions $h \in \mathcal{H}$ that satisfy Assumption 1. Define the following differential operators:

- $\gamma(h) = h'^2/h''$.
- $\iota(h)(p) = ph''(p)/(-h')$ for every $p > 0$.
- $\rho(h) = h/\gamma$.

We will need the following technical lemma:

Lemma III. *If $h \in \mathcal{H}$, then:*

(a) $\lim_{p\to\infty} ph'(p) = \lim_\infty h' = 0$.

Moreover, if $h \in \mathcal{H}^\iota$, then:

(b) There exists a unique scalar $\underline{p}(h) \geq 0$ such that for every $p > 0$, $\iota(h)(p) > 1$ if and only if $p > \underline{p}(h)$. Moreover, $\left(\iota(h)\right)'(p) \geq 0$ for all $p > \underline{p}(h)$.

(c) $\bar{\mu}(h) \equiv \lim_{p\to\infty} \iota(h)(p) > 1$.

(d) For every $p > \underline{p}(h)$, $\left(\gamma(h)\right)'(p) < 0$.

(e) $\lim_{p\to\infty} \gamma(h)(p) = 0$.

(f) If $\lim_\infty h = 0$ and $\bar{\mu}(h) < \infty$, then $\lim_{p\to\infty} \rho(h)(p) = \frac{\bar{\mu}(h)}{\bar{\mu}(h)-1}$.

Proof. In the following, we drop argument h from functions γ, ι, ρ, $\underline{p}$ and $\bar{\mu}$ to ease notation.

(a) We first show that $\lim_{p\to\infty} ph'(p)$ exists. By the fundamental theorem of calculus, for every $p > 0$,

$$h(p) = h(1) + \int_1^p h'(x)dx = h(1) + ph'(p) - h'(1) - \int_1^p xh''(x)dx,$$

where the second line was obtained by integrating by parts. Therefore, $ph'(p) = h(p) - h(1) + h'(1) + \int_1^p xh''(x)dx$. Since h is positive and decreasing, it has a finite limit at ∞. We now show that $\int_1^p xh''(x)dx$ also has a limit at infinity. Since h is log-convex, $(\log h)'' = \frac{h''h-h'^2}{h^2} \geq 0$. It follows that $h'' \geq 0$. Therefore, function $p \mapsto \int_1^p xh''(x)dx$ is non-decreasing, and that function has a limit at infinity. It follows that $\lim_{p\to\infty} ph'(p)$ exists. Since $h' < 0$, that limit is non-positive.

Assume for a contradiction that $\lim_{p\to\infty} ph'(p) < 0$. Then, there exist $\varepsilon_0 > 0$ and $p_0 > 0$ such that $ph'(p) \leq -\varepsilon_0$ for all $p \geq p_0$. Rewrite this inequality as $h'(p) \leq -\varepsilon_0/p$, and integrate it between p^0 and p to get

$$h(p) - h(p_0) \leq -\varepsilon_0 \log\left(\frac{p}{p_0}\right) \underset{p\to\infty}{\longrightarrow} -\infty.$$

Therefore, $\lim_\infty h = -\infty$. This contradicts the assumption that $h > 0$.

Therefore, $\lim_{p\to\infty} ph'(p) = 0$, and $\lim_\infty h = 0$.

(b) Assume for a contradiction that $\iota(p) \leq 1$ for all $p > 0$. Then, for all $p > 0$, $ph''(p) + h'(p) \leq 0$, i.e., $\frac{d}{dp}(ph'(p)) \leq 0$. It follows that $ph'(p) \leq h'(1)$ for all $p \geq 1$. Taking the limit as p goes to infinity and using point (a), we obtain that $h'(1) \geq 0$, a contradiction.

Therefore, there exists $\hat{p} > 0$ such that $\iota(\hat{p}) > 1$, and

$$\underline{p} \equiv \inf\{p \in \mathbb{R}_{++} : \quad \iota(p) > 1\} < \infty.$$

We prove two claims:

Claim 1: $\underline{p} \notin \{p > 0 : \ \iota(p) > 1\}$.

If $\underline{p} = 0$, then this is obvious. If instead $\underline{p} > 0$, then the claim follows immediately from the continuity of ι.

Claim 2: $\iota(y) \geq \iota(x)$ whenever $0 < x < y$ and $\iota(x) > 1$.

Assume for a contradiction that $\iota(y) < \iota(x)$. Put $S = \{z \in [x, y] : \ \iota(z) \leq 1\}$. If S is empty, then $\iota(z) > 1$ for every $z \in [x, y]$. Since $h \in \mathcal{H}^\iota$, $\iota'(z) \geq 0$ for every $z \in [x, y]$, and ι is non-decreasing on interval $[x, y]$. It follows that $\iota(y) \geq \iota(x)$, which is a contradiction.

Next, assume that S is not empty. Then, $\hat{y} \equiv \inf S \in [x, y]$. Moreover, by continuity of ι, and since $\iota(x) > 1$, $\iota(\hat{y}) = 1$. In addition, $\iota(z) > 1$ for every $z \in [x, \hat{y})$. Using the same reasoning as above, it follows that

$$1 = \iota_k(\hat{y}) \geq \iota_k(x) > 1,$$

which is a contradiction.

Combining Claims 1 and 2, it follows that $\{x > 0 : \iota(x) > 1\} = \big(\underline{p}, \infty\big)$, and that ι is non-decreasing on $\big(\underline{p}, \infty\big)$, which proves point (b).

(c) Since ι is monotone on $(\underline{p}, \infty)$, $\bar{\mu}$ exists. Assume for a contradiction that $\bar{\mu} \leq 1$. Then, by monotonicity, $\iota(p) \leq \bar{\mu} \leq 1$ for every $p > \underline{p}$. This contradicts point (b).

(d) Let $p > \underline{p}$. Notice that

$$\gamma(p) = \frac{-h'(p)}{ph''(p)} \left(p(-h'(p))\right) = \frac{-ph'(p)}{\iota(p)}.$$

Therefore,

$$\begin{aligned}\gamma'(p) &= \frac{1}{(\iota(p))^2} \left(- \left(ph''(p) + h'(p)\right) \times \iota(p) + \iota'(p) \times ph'(p)\right), \\ &= \frac{1}{(\iota(p))^2} \left(-h'(p) \left(1 - \iota(p)\right) \iota(p) + \iota'(p)ph'(p)\right) < 0,\end{aligned}$$

as $\iota' \geq 0$ and $\iota(p) > 1$ for all $p > \underline{p}$.

(e) The result follows immediately from the fact that $\gamma(p) = -ph'(p)/\iota(p)$ (see above), $\lim_{p\to\infty} ph'(p) = 0$ (point (a)), and $\lim_\infty \iota > 0$ (point (c)).

(f) Suppose $\bar{\mu} < \infty$ and $\lim_\infty h = 0$. For all $p > \underline{p}$,

$$\rho(p) = \frac{h(p)h''(p)}{(h'(p))^2} = \frac{ph''(p)}{-h'(p)} \frac{h(p)}{-ph'(p)} = \iota(p)\frac{h(p)}{-ph'(p)}.$$

By assumption, $\lim_\infty h = 0$. By point (a), $\lim_{p\to\infty} -ph'(p) = 0$. Moreover,

$$\lim_{p\to\infty} \frac{\frac{d}{dp}h(p)}{\frac{d}{dp}\left(-ph'(p)\right)} = \lim_{p\to\infty} \frac{h'(p)}{-h'(p) - ph''(p)} = \lim_{p\to\infty} \frac{1}{\iota(p) - 1} = \frac{1}{\bar{\mu} - 1}.$$

Therefore, by L'Hospital's rule, $\lim_{p\to\infty} \frac{h(p)}{-ph'(p)} = \frac{1}{\bar{\mu}-1}$, and $\lim_\infty \rho = \frac{\bar{\mu}}{\bar{\mu}-1}$. □

II.2 About the (Log)-Supermodularity of Payoff Functions

Fix a pricing game $((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$, and let $f \in \mathcal{F}$ such that $|f| \geq 2$. Fix a vector of prices for firm f's rivals $(p_j)_{j\in\mathcal{N}\setminus f}$, and let $H^0 = \sum_{j\notin f} h_j(p_j)$. We introduce the following notation: $\nu_i(p_i) = \frac{p_i - c_i}{p_i}\iota_i(p_i)$ for every i and $p_i > 0$.

We first show that Π^f is neither supermodular nor submodular in $(p_j)_{j\in f}$. Let $i \neq k$ in f.

$$\frac{\partial^2 \Pi^f}{\partial p_i \partial p_k} = \frac{\partial}{\partial p_k}\left(\frac{-h_i'(p_i)}{H}\left(1-\nu_i(p_i)+\Pi^f(p)\right)\right),$$
$$= -h_i'\left(\frac{-h_k'}{H^2}(1-\nu_i+\Pi^f)+\frac{1}{H}\frac{-h_k'}{H}\left(1-\nu_k+\Pi^f\right)\right),$$
$$= \frac{h_i'h_k'}{H^2}\left((1-\nu_i+\Pi^f)+(1-\nu_k+\Pi^f)\right), \tag{ii}$$

where we have used the expression of marginal profit derived in equation (4).

Assume in addition that firm f's profile of prices satisfies the constant ι-markup property. Then, equation (ii) can be simplified as follows:

$$\frac{\partial^2 \Pi^f}{\partial p_i \partial p_k} = \frac{2h_i'h_k'}{H^2}\left(1-\mu^f+\frac{1}{H}\mu^f\sum_{j\in f}\gamma_j(r_j(\mu^f))\right),$$
$$= -\frac{2h_i'h_k'}{H^3}\underbrace{\left((\mu^f-1)\left(H^0+\sum_{j\in f}h_j(r_j(\mu^f))\right)-\mu^f\sum_{j\in f}\gamma_j(r_j(\mu^f))\right)}_{\equiv\phi(\mu^f)}.$$

We have shown in the proof of Lemma F that $\phi(\mu^f)$ is strictly positive when μ^f is large, and strictly negative when μ^f is small. It follows that Π^f is neither supermodular nor submodular in $(p_j)_{j\in f}$.

Next, we show that Π^f is neither log-supermodular nor log-submodular in $(p_j)_{j\in f}$. Let $i \neq k$ in f.

$$\frac{\partial^2 \log \Pi^f}{\partial p_i \partial p_k} = \frac{\partial}{\partial p_k}\left(\frac{-h_i'-(p_i-c_i)h_i''}{\sum_{j\in f}(p_j-c_j)(-h_j')}+\frac{-h_i'}{H}\right),$$
$$= -\frac{\left(-h_i'-(p_i-c_i)h_i''\right)\left(-h_k'-(p_k-c_k)h_k''\right)}{\left(\sum_{j\in f}(p_j-c_j)(-h_j')\right)^2}+\frac{h_i'h_k'}{H^2},$$
$$= \frac{h_i'h_k'}{H^2}\left(1-\frac{(\nu_i-1)(\nu_k-1)}{(\Pi^f)^2}\right).$$

Again, if firm f's profile of prices has the constant ι-markup property, then

$$\frac{\partial^2 \Pi^f}{\partial p_i \partial p_k} = \frac{h_i'h_k'}{H^2}\left(1-\left(\frac{\mu^f-1}{\Pi^f}\right)^2\right).$$

Note that

$$\frac{\mu^f-1}{\Pi^f} = 1+\frac{\phi(\mu^f)}{\mu^f\sum_{j\in f}\gamma_j(r_j(\mu^f))}.$$

Let μ^{f*} be the unique solution of equation $\phi(\mu^f) = 0$. Then, by continuity, for μ^f close enough to μ^{f*} and strictly below μ^{f*}, $(\mu^f - 1)/\Pi^f \in (0,1)$, and, therefore, $\partial^2 \Pi^f / \partial p_i \partial p_k > 0$. For μ^f close enough to μ^{f*} and strictly above μ^{f*}, $(\mu^f - 1)/\Pi^f > 1$, and, therefore, $\partial^2 \Pi^f / \partial p_i \partial p_k < 0$. Therefore, Π^f is neither log-supermodular nor log-submodular in $(p_j)_{j \in f}$.

III Assumption 1 and First-Order Conditions

The goal of this section is to formalize and prove our statement that Assumption 1 is the weakest assumption under which an approach based on first-order conditions is valid.

III.1 Definitions and Statement of the Theorem

We first define a multiproduct firm as a collection of products, along with a constant unit cost for each product:

Definition 1. *A multiproduct firm is a pair $((h_j)_{j \in \mathcal{N}}, (c_j)_{j \in \mathcal{N}})$, where $\mathcal{N} = \{1, \ldots, n\}$ is a finite and non-empty set, and for every $j \in \mathcal{N}$, $h_j \in \mathcal{H}$, and $c_j > 0$.*[2] *The profit function associated with multi-product firm M is:*

$$\Pi(M)\left(p, H^0\right) = \sum_{k \in \mathcal{N}} (p_k - c_k) \frac{-h_k'(p_k)}{\sum_{j \in \mathcal{N}} h_j(p_j) + H^0}, \quad \forall p \in \mathbb{R}^{\mathcal{N}}_{++}, \ \forall H^0 > 0.$$

As in the paper, H^0 represents the value of the outside option. Our goal is to derive conditions under which profit function $\Pi(M)(\cdot, H^0)$ is well-behaved.

In the following, it will be useful to study multiproduct firms that can be constructed from a set of products (i.e., a set of indirect subutility functions) smaller than $\mathcal{H}$:

Definition 2. *The set of multiproduct firms that can be constructed from set $\mathcal{H}' \subseteq \mathcal{H}$ is:*

$$\mathcal{M}(\mathcal{H}') = \bigcup_{n \in \mathbb{N}_{++}} \left(\mathcal{H}'^n \times \mathbb{R}^n_{++}\right).$$

We can now define well-behaved multiproduct firms and well-behaved sets of products:

Definition 3. *We say that multiproduct firm $M \in \mathcal{M}(\mathcal{H})$ is well-behaved if for every $(p, H^0) \in \mathbb{R}^{n+1}_{++}$, $\nabla_p \Pi(M)(p, H^0) = 0$ implies that p is a local maximizer of $\Pi(M)(., H^0)$. We say that product set $\mathcal{H}' \subseteq \mathcal{H}$ is well-behaved if every $M \in \mathcal{M}(\mathcal{H}')$ is well-behaved.*

Put differently, a set of products is well-behaved if for every multiproduct firm that can be constructed from this set, for every value the outside option H^0 can take, first-order conditions are sufficient for local optimality. In the following, we look for the "largest" well-behaved set of products, where the meaning of "large" will be made more precise shortly.

[2] Recall from Section II that $\mathcal{H}$ is the set of strictly decreasing, $\mathcal{C}^3$ and log-convex functions from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$.

We define the set of CES products as follows:

$$\mathcal{H}^{CES} = \left\{ h \in \mathcal{H} : \quad \exists (a, \sigma) \in \mathbb{R}_{++} \times (1, \infty) \text{ s.t. } \forall p > 0,\ h(p) = ap^{1-\sigma} \right\}.$$

We have shown in the paper that $\mathcal{H}^{CES} \subseteq \mathcal{H}^{\iota}$.

We are now in a position to state our theorem:

Theorem II. *$\mathcal{H}^{\iota}$ is the largest (in the sense of set inclusion) set $\mathcal{H}' \subseteq \mathcal{H}$ such that $\mathcal{H}^{CES} \subseteq \mathcal{H}'$ and $\mathcal{H}'$ is well-behaved.*

In words, $\mathcal{H}^{\iota}$ is the largest set of products that contains CES products and that is well-behaved. Rephrasing this result in terms of pricing games, this means that pricing games based on sets of products larger than $\mathcal{H}^{\iota}$ are not well-behaved, and that an aggregative games approach based on first-order conditions is not valid.

III.2 Proof of Theorem II

We first make the dependence of function ν_k on marginal cost c_k explicit by writing $\nu_k(p_k, c_k) \equiv \frac{p_k - c_k}{p_k} \iota_k(p_k)$. (Function ν_k was first defined in Section II.) Note that

$$\frac{\partial \nu_k}{\partial p_k} = \frac{c_k}{p_k^2} \iota_k(p_k) + \frac{p_k - c_k}{p_k} \iota_k'(p_k). \tag{iii}$$

In addition, since $\iota_k(p_k) = p_k \frac{-h_k'(p_k)}{\gamma_k(p_k)}$, we also have that

$$\frac{\partial \nu_k}{\partial p_k} = \frac{\left(\nu_k(p_k, c_k) - 1\right) h_k'(p_k) - \nu_k(p_k, c_k) \gamma_k'(p_k)}{\gamma_k(p_k)}. \tag{iv}$$

Differentiating the monopolist's profit with respect to p_k, we obtain:

$$\begin{aligned} \frac{\partial \Pi\left(M\right)}{\partial p_k} &= \frac{-h_k'(p_k)}{H} \left(1 - \frac{p_k - c_k}{p_k} p_k \frac{-h_k''(p_k)}{-h_k'(p_k)} + \sum_{j \in \mathcal{N}} (p_j - c_j) \frac{-h_j'(p_j)}{H} \right), \\ &= \frac{-h_k'(p_k)}{H} \left(1 - \nu_k(p_k, c_k) + \sum_{j \in \mathcal{N}} \nu_j(p_j, c_j) \frac{\gamma_j(p_j)}{H} \right), \end{aligned} \tag{v}$$

where $H = \sum_{j \in \mathcal{N}} h_j(p_j) + H^0$. Therefore, if the first-order conditions hold at price vector p, then, for every k in $\mathcal{N}$,

$$\nu_k(p_k, c_k) = 1 + \sum_{j \in \mathcal{N}} \nu_j(p_j, c_j) \frac{\gamma_j(p_j)}{H}. \tag{vi}$$

Since the right-hand side of the above equation does not depend on the identity of product k, it follows that p satisfies the common-ι markup property:

$$\nu(p_i, c_i) = \nu(p_j, c_j), \quad \forall i, j \in \mathcal{N}.$$

This allows us to rewrite the first-order condition for product k as follows:

$$\nu_k(p_k, c_k)\left(1 - \sum_{j\in\mathcal{N}} \frac{\gamma_j(p_j)}{H}\right) = 1. \tag{vii}$$

Since we are interested in the sufficiency of first-order conditions for local optimality, we need to calculate the Hessian of the monopolist's profit function. This is done in the following lemma:

Lemma IV. *Let $M \in \mathcal{M}(\mathcal{H})$, $p >> 0$ and $H^0 > 0$. If $\nabla_p \Pi(M)(p, H^0) = 0$, then the Hessian of $\Pi(M)(., H^0)$, evaluated at price vector p, is diagonal, with typical diagonal element*

$$\frac{h'_k(p_k)}{H^0 + \sum_{j\in\mathcal{N}} h_j(p_j)} \frac{\partial \nu_k}{\partial p_k}(p_k, c_k).$$

Proof. Let $M = \left((h_j)_{j\in\mathcal{N}}, (c_j)_{j\in\mathcal{N}}\right) \in \mathcal{M}(\mathcal{H})$. Let $p >> 0$ and $H^0 > 0$, and suppose that $\nabla_p \Pi(M)(p, H^0) = 0$. For every $1 \leq k \leq n$,

$$\begin{aligned}
\frac{\partial^2 \Pi(M)}{\partial p_k^2} &= \frac{-h'_k}{H}\left(-\frac{\partial \nu_k}{\partial p_k} + \frac{1}{H}\left(\frac{\partial \nu_k}{\partial p_k}\gamma_k + \nu_k \gamma'_k - \nu_k \frac{\sum_{j\in\mathcal{N}} \gamma_j}{H} h'_k\right)\right), \\
&= \frac{-h'_k}{H}\left(-\frac{\partial \nu_k}{\partial p_k} + \frac{1}{H}\left(\frac{\partial \nu_k}{\partial p_k}\gamma_k + \nu_k \gamma'_k - (\nu_k - 1) h'_k\right)\right), \\
&= \frac{-h'_k}{H}\left(-\frac{\partial \nu_k}{\partial p_k} + \frac{1}{H}\left(\frac{\partial \nu_k}{\partial p_k}\gamma_k - \frac{\partial \nu_k}{\partial p_k}\gamma_k\right)\right), \\
&= \frac{h'_k}{H}\frac{\partial \nu_k}{\partial p_k}.
\end{aligned}$$

where the first line follows from differentiating equation (v) with respect to p_k and using the fact that $\partial \Pi(M)/\partial p_k = 0$, the second line follows from equation (vii), and the third line follows from equation (iv). Using the same method, we find that all the off-diagonal elements of the Hessian matrix are equal to zero, which proves the lemma. □

The following lemma is an immediate consequence of Lemma IV and equation (iii):

Lemma V. *Set $\mathcal{H}^\iota$ is well-behaved.*

Proof. Let $M = \left((h_j)_{j\in\mathcal{N}}, (c_j)_{j\in\mathcal{N}}\right) \in \mathcal{M}(\mathcal{H})$. Let $p >> 0$ and $H^0 > 0$, and suppose that $\nabla_p \Pi(M)(p, H^0) = 0$. Then, by equation (vii), and by log-convexity of h_j for every j, $\nu_k(p_k, c_k) > 1$ for every $1 \leq k \leq n$. It follows that $\iota_k(p_k) > 0$ and $p_k > c_k$ for every k. Therefore, by equation (iii) and since $h_k \in \mathcal{H}^\iota$, $\partial \nu_k / \partial p_k > 0$. By Lemma IV, the Hessian of $\Pi(M)(., H^0)$ evaluated at price vector p is therefore negative definite. Therefore, the local second-order conditions hold, p is a local maximizer of $\Pi(M)(., H^0)$, M is well-behaved, and $\mathcal{H}^\iota$ is well-behaved. □

The next step is to rule out products that are not in $\mathcal{H}^\iota$. This is done in the following lemma:

Lemma VI. *Let $h \in \mathcal{H} \backslash \mathcal{H}^\iota$. Then, $\mathcal{H}^{CES} \cup \{h\}$ is not well-behaved.*

Proof. Since $h \notin \mathcal{H}^\iota$, there exists $\hat{p} > 0$ such that $\iota(\hat{p}) > 1$ and $\iota'(\hat{p}) < 0$. Our goal is to construct a two-product firm $M = ((h_1, h_2), (c_1, c_2))$, a price vector $(p_1, p_2) \in \mathbb{R}^2_{++}$ and an $H^0 > 0$ such that $\nabla_p \Pi(M)((p_1, p_2), H^0) = 0$ and $\frac{\partial \nu_1}{\partial p_1}(p_1, c_1) < 0$. We begin by setting $h_1 = h$ and $p_1 = \hat{p}$. We will tweak h_2, p_2, c_1, c_2 and H^0 along the way.

Since $\iota_1'(p_1) < 0$, equation (iii) implies that there exists $\bar{c} \in (0, p_1)$ such that $\frac{\partial \nu_1}{\partial p_1}(p_1, c_1) < 0$ whenever $c_1 < \bar{c}$.

For every $s \in (1, \iota_1(p_1))$, there exists a unique $C_1(s) \in (0, p_1)$ such that

$$\frac{p_1 - C_1(s)}{p_1} \frac{\iota_1(p_1)}{s} = 1. \tag{viii}$$

$C_1(\cdot)$ is continuous and $\lim_{s \to \iota_1(p_1)} C_1(s) = 0$. In particular, there exists $\underline{s} \in (1, \iota_1(p_1))$ such that $C_1(s) \in (0, \bar{c})$ whenever $s \in (\underline{s}, \iota_1(p_1))$. It follows that, when $s \in (\underline{s}, \iota_1(p_1))$, condition (viii) holds and $\frac{\partial \nu_1}{\partial p_1}(p_1, C_1(s)) < 0$.

Let $\sigma \in (\underline{s}, \iota_1(p_1))$, and $h_2(p_2) = p_2^{1-\sigma}$ for all $p_2 > 0$. Recall that $\iota_2(p_2) = \sigma$ and $\gamma_2(p_2) = \frac{\sigma-1}{\sigma} h_2(p_2)$ for all $p_2 > 0$.

For every $H^0 > 0$, define the following function:

$$\phi(x) = 1 - \frac{\gamma_1(p_1) + \frac{\sigma-1}{\sigma} x}{h_1(p_1) + x + H^0}, \ \forall x > 0.$$

Notice that $\lim_\infty \phi = \frac{1}{\sigma}$. Moreover,

$$\phi'(x) = \frac{\gamma_1(p_1) - \frac{\sigma-1}{\sigma}\left(h_1(p_1) + H^0\right)}{\left(h_1(p_1) + x + H^0\right)^2}.$$

Choose some H^0 such that $\gamma_1(p_1) - \frac{\sigma-1}{\sigma}\left(h_1(p_1) + H^0\right) < 0$. Then, $\phi'(x) < 0$ for all $x > 0$. Therefore, $\phi(x) > \frac{1}{\sigma}$ for all $x > 0$.

Let $(p_2, c_2) \in \mathbb{R}^2_{++}$. The first-order condition for product 2 can be written as follows:

$$\frac{p_2 - c_2}{p_2} \sigma \left(1 - \frac{\gamma_1(p_1) + \gamma_2(p_2)}{h_1(p_1) + h_2(p_2) + H^0}\right) = 1,$$

or, equivalently,

$$\frac{p_2 - c_2}{p_2} \times \underbrace{\sigma \phi\left(p_2^{1-\sigma}\right)}_{>1, \text{ since } \phi(x) > 1/\sigma} = 1.$$

Therefore, for every $p_2 > 0$, there exists a unique $C_2(p_2) \in (0, p_2)$ such that the first-order condition for product 2 holds.

The first-order condition for product 1 can be written as follows:

$$\frac{p_1 - c_1}{p_1} \frac{\iota_1(p_1)}{\phi\left(p_2^{1-\sigma}\right)^{-1}} = 1.$$

Since $\phi\left(p_2^{1-\sigma}\right)^{-1} \underset{p_2 \to 0^+}{\longrightarrow} \sigma$ and $\sigma \in (\underline{s}, \iota_1(p_1))$, there exists $P_2 > 0$ such that $\phi\left(P_2^{1-\sigma}\right)^{-1} \in (\underline{s}, \iota_1(p_1))$. Put $c_1 = C_1\left(\phi\left(P_2^{1-\sigma}\right)^{-1}\right)$. Then, the first-order condition for product 1 holds, $c_1 \in (0, \bar{c})$, and therefore, $\frac{\partial \nu_1}{\partial p_1}(p_1, c_1) < 0$.

To summarize, we have constructed a multi-product firm $M = ((h_1, h_2), (c_1, c_2))$ with $h_1 = h$, $h_2(x) = x^{1-\sigma}$, $c_1 = C_1\left(\phi\left(P_2^{1-\sigma}\right)^{-1}\right)$ and $c_2 = C_2(P_2)$, an $H^0 > 0$ and a price vector $(p_1, p_2) = (\hat{p}, P_2)$ such that $\nabla_p \Pi(M)\left((p_1, p_2), H^0\right) = 0$ and $\frac{\partial \nu_1}{\partial p_1}(p_1, c_1) < 0$. By Lemma IV, the Hessian matrix of $\Pi(M)(\cdot, H^0)$ evaluated at price vector (p_1, p_2) has a strictly positive eigenvalue. Therefore, (p_1, p_2) is not a local maximizer of $\Pi(M)(\cdot, H^0)$, and multi-product firm M is not well-behaved. It follows that $\mathcal{H}^{CES} \cup \{h\}$ is not well-behaved. $\square$

Combining Lemmas V and VI proves Theorem II.

III.3 A Remark on Single-Product Firms

We close this section by noting that multiproduct-firms are special, in the sense that, compared to single-product firms, they require strictly stronger restrictions on the set of admissible products to be well-behaved. This statement is formalized in the following proposition:

Proposition I. *Let $h \in \mathcal{H}$, $c > 0$ and $M = (h, c)$. The following assertions are equivalent:*

(i) Firm M is well-behaved.

(ii) For every $p > 0$ such that $\iota(h)(p) > 1$, $(\iota(h))'(p) \geq 0$ or $(\rho(h))'(p) \geq 0$.

Proof. Let $h \in \mathcal{H}$, $c > 0$ and $M = (h, c)$. With single-product firms, first-order condition (vii) can be simplified as follows:

$$\nu\left(1 - \frac{\gamma}{h + H^0}\right) = 1. \tag{ix}$$

By Lemma IV, $\partial^2 \Pi(M)/\partial p^2$ has the same sign as $\partial \nu / \partial p$ whenever condition (ix) holds.

Assume that (ii) holds. We want to show that, for every $(p, c, H^0) \in \mathbb{R}^3_{++}$, $\partial \nu(p, c)/\partial p > 0$ whenever condition (ix) holds. Let $p > 0$. If $\iota(p) \leq 1$, then for every $c, H^0 > 0$,

$$\nu\left(1 - \frac{\gamma}{h + H^0}\right) < 1,$$

so there is nothing to prove. Next, assume that $\iota(p) > 1$. For every $c > 0$, $\partial \nu / \partial p$ is given by equation (iii). If $\iota'(p) \geq 0$, then $\partial \nu(p, c)/\partial p > 0$ for every $H^0 > 0$ and $0 < c \leq p$.

In particular, $\partial\nu(p,c)/\partial p > 0$ when condition (ix) holds. (Recall that, by log-convexity, $\gamma < h + H^0$.)

Assume instead that $\iota'(p) < 0$. Then, since (ii) holds, $\rho'(p) \geq 0$. Notice that

$$\frac{\rho'}{\rho} = \left(\log\left(\frac{h\iota}{p(-h')}\right)\right)' = \frac{h'}{h} + \frac{\iota'}{\iota} - \frac{1}{p} + \frac{h''}{-h'}.$$

It follows that

$$p\frac{\rho'}{\rho} = p\frac{\iota'}{\iota} - p\frac{-h'}{h} - 1 + \iota = p\frac{\iota'}{\iota} - \frac{\iota}{\rho} - 1 + \iota = p\frac{\iota'}{\iota} + \iota\left(1 - \frac{1}{\rho}\right) - 1.$$

Since $\iota' < 0$ and $\rho' \geq 0$, it follows that $\iota\left(1 - \frac{1}{\rho}\right) - 1 > 0$.

Since $\iota(p) > 1$, we have that, for every $H^0 > 0$, there exists a unique $c\left(H^0\right)$ such that condition (ix) holds. This $c\left(H^0\right)$ is given by:

$$c\left(H^0\right) = p\left(1 - \frac{1}{\iota\left(1 - \frac{\gamma}{h+H^0}\right)}\right). \tag{x}$$

Since $\iota\left(1 - \frac{1}{\rho}\right) - 1 > 0$, $c\left(H^0\right) \in (0,p)$ for every $H^0 > 0$. Notice also that $c'\left(H^0\right) > 0$. All we need to do now is check that

$$\frac{\partial\nu}{\partial p}\left(p, c\left(H^0\right)\right) = \frac{c\left(H^0\right)}{p^2}\iota + \frac{p - c\left(H^0\right)}{p}\iota'$$

is strictly positive for every $H^0 > 0$. Since the right-hand side is strictly increasing in $c\left(H^0\right)$ and $c'\left(H^0\right) > 0$, this boils down to checking that $\partial\nu\left(p, c(0)\right)/\partial p \geq 0$:

$$\begin{aligned}
\frac{\partial\nu}{\partial p}\left(p, c(0)\right) &= \frac{\iota}{p}\left(\frac{c(0)}{p}\iota + \frac{p - c(0)}{p}p\frac{\iota'}{\iota}\right), \\
&= \frac{\iota}{p}\left(\left(1 - \frac{1}{\iota\left(1 - \frac{1}{\rho}\right)}\right) + \frac{1}{\iota\left(1 - \frac{1}{\rho}\right)}p\frac{\iota'}{\iota}\right), \\
&= \frac{1}{p\left(1 - \frac{1}{\rho}\right)}\left(\iota\left(1 - \frac{1}{\rho}\right) - 1 + p\frac{\iota'}{\iota}\right), \\
&= \frac{\rho'}{\rho - 1},
\end{aligned}$$

which is indeed non-negative. Therefore, (i) holds.

Conversely, suppose that (ii) does not hold. There exists $p > 0$ such that $\iota(p) > 1$, $\iota'(p) < 0$ and $\rho'(p) < 0$. We distinguish two cases. Assume first that $\iota\left(1 - \frac{1}{\rho}\right) - 1 \geq 0$. Then, the

$c\left(H^0\right)$ defined in equation (x) satisfies $c(H^0) \in (0, p)$ and

$$\frac{p - c\left(H^0\right)}{p} \iota \left(1 - \frac{\gamma}{h + H^0}\right) = 1$$

for every $H^0 > 0$. In addition, as proven above,

$$\frac{\partial \nu}{\partial p}\left(p, c(0)\right) = \frac{\rho'}{\rho - 1} < 0.$$

By continuity, there exists $\varepsilon > 0$ such that $\frac{\partial \nu}{\partial p}\left(p, c(\varepsilon)\right) < 0$. It follows that $\frac{\partial \Pi(M)}{\partial p}(p, \varepsilon) = 0$ and $\frac{\partial^2 \Pi(M)}{\partial p^2}(p, \varepsilon) > 0$. Therefore, M is not well-behaved.

Next, assume that $\iota\left(1 - \frac{1}{\rho}\right) - 1 < 0$. Then, there exists $H^0 > 0$ such that $c\left(H^0\right) = 0$. Notice that $\frac{\partial \nu}{\partial p}\left(p, 0\right) = \iota'(p) < 0$. Therefore, by continuity of $\partial \nu / \partial p$ and $c(.)$, for $\varepsilon > 0$ small enough,

$$\frac{\partial \nu}{\partial p}\left(p, c\left(H^0 + \varepsilon\right)\right) < 0,$$

and $c\left(H^0 + \varepsilon\right) > 0$. Therefore, multiproduct firm $\left(h, c\left(H^0 + \varepsilon\right)\right)$ is not well-behaved. □

IV Additive Aggregation and Demand Systems

IV.1 Characterization Result

Let $\mathcal{G} = (\mathcal{I}, (A_i)_{i \in \mathcal{I}}, (\pi_i)_{i \in \mathcal{I}})$ be a normal-form game. Suppose that each action space A_i is a cartesian product of intervals. We say that game G is aggregative with additive and smooth aggregation if there exist collections of $\mathcal{C}^2$ functions $(\psi_j)_{j \in \mathcal{I}}$ and $(\phi_j)_{j \in \mathcal{I}}$ such that for every $a = (a_j)_{j \in \mathcal{I}} \in \prod_{j \in \mathcal{I}} A_j$ and $i \in \mathcal{I}$,

$$\pi_i(a) = \phi_i\left(a_i, \sum_{j \in \mathcal{I}} \psi_j(a_j)\right).$$

We prove the following proposition:

Proposition II. *Let $D : \mathbb{R}^{\mathcal{N}}_{++} \longrightarrow \mathbb{R}^{\mathcal{N}}$ be a $\mathcal{C}^2$ and quasi-linear demand system, where $\mathcal{N}$ is a finite set containing at least three elements. Suppose that D satisfies Slutsky symmetry, and that $\partial D_i(p)/\partial p_j \neq 0$ for every $i \neq j$ and $p >> 0$. The following assertions are equivalent:*

(i) Any multiproduct-firm pricing game based on D is aggregative with smooth and additive aggregation.

(ii) There exist $\mathcal{C}^3$ functions Ψ, $(g_i)_{i\in\mathcal{N}}$, and $(h_i)_{i\in\mathcal{N}}$ such that

$$D_i(p) = -g_i'(p_i) - h_i'(p_i)\Psi'\left(\sum_{j\in\mathcal{N}} h_j(p_j)\right), \quad \forall i \in \mathcal{N},\ \forall p >> 0. \tag{xi}$$

Proof. It is obvious that (ii) implies (i). Assume that (i) holds, and consider the pricing game with firm partition $\{\{i\}\}_{i\in\mathcal{N}}$ and zero marginal cost. Since (i) holds, there exist $\mathcal{C}^2$ functions $\phi_i(p_i, H)$ and $h_i(p_i)$ for every i such that, for every $i \in \mathcal{N}$, the profit of firm $\{i\}$ is given by:

$$\Pi^{\{i\}}(p) = \phi_i\left(p_i, \sum_{j\in\mathcal{N}} h_j(p_j)\right) = p_i D_i(p).$$

It follows that

$$D_i(p) = \frac{1}{p_i}\phi_i\left(p_i, \sum_{j\in\mathcal{N}} h_j(p_j)\right) \equiv f_i\left(p_i, \sum_{j\in\mathcal{N}} h_j(p_j)\right), \quad \forall i.$$

Since $\partial D_i/\partial p_j(p) \neq 0$, it follows that $h_i'(p_i) \neq 0$ for every p_i, and $\partial f_i(p_i, H)/\partial H \neq 0$ for every p_i and H.

By Slutsky symmetry, for every $i \neq j$,

$$h_j' \frac{\partial f_i}{\partial H}(p_i, H) = \frac{\partial D_i}{\partial p_j} = \frac{\partial D_j}{\partial p_i} = h_i' \frac{\partial f_j}{\partial H}(p_j, H). \tag{xii}$$

Next, we differentiate the Slutsky condition with respect to p_k, $k \neq i, j$:

$$h_j' h_k' \frac{\partial^2 f_i}{\partial H^2} = h_i' h_k' \frac{\partial^2 f_j}{\partial H^2}.$$

Since $h_k' \neq 0$, it follows that

$$h_j' \frac{\partial^2 f_i}{\partial H^2} = h_i' \frac{\partial^2 f_j}{\partial H^2}. \tag{xiii}$$

Next, differentiate the Slutsky condition with respect to p_i:

$$h_j' \frac{\partial^2 f_i}{\partial p_i \partial H} + h_j' h_i' \frac{\partial^2 f_i}{\partial H^2} = h_i'' \frac{\partial f_j}{\partial H} + h_i'^2 \frac{\partial^2 f_j}{\partial H^2}.$$

Therefore, using equation (xiii),

$$h_j' \frac{\partial^2 f_i}{\partial p_i \partial H} = h_i'' \frac{\partial f_j}{\partial H}.$$

Next, we use equation (xii) to eliminate $\partial f_j/\partial H$ and h_j'. This yields:

$$\frac{\frac{\partial^2 f_i}{\partial p_i \partial H}(p_i, H)}{\frac{\partial f_i}{\partial H}(p_i, H)} = \frac{h_i''}{h_i'}.$$

The above condition must hold for every (p_i, H) in the domain of f_i. Note that it depends only on p_i and H (and not on p_j for $j \neq i$). Integrating this partial differential equation, we obtain:

$$\frac{\partial f_i}{\partial H}(p_i, H) = h_i'(p_i)\lambda_i(H),$$

where $\lambda_i(H)$ is a constant of integration. Integrating once more, we obtain:

$$f_i(p_i, H) = h_i'(p_i)\Lambda_i(H) + g_i'(p_i),$$

where Λ_i is an anti-derivative of λ_i, and g_i' is a constant of integration. Therefore,

$$D_i(p) = h_i'(p_i)\Lambda_i\left(\sum_{j \in \mathcal{N}} h_j(p_j)\right) + g_i'(p_i), \quad \forall i.$$

Next, we use Slutsky symmetry one more time:

$$h_i' h_j' \Lambda_i'(H) = h_i' h_j' \Lambda_j'(H).$$

Therefore, Λ_i and Λ_j differ by an additive constant, which we can safely ignore (or, rather, incorporate in the g_i' functions). It follows that (ii) holds. □

IV.2 The Generalized Common ι-Markup Property

Fix a pricing game based on demand system (xi). It is easy to show that a generalized form of the constant ι-markup property still holds. Let $f \in \mathcal{F}$ and $i \in f$. Then,

$$\frac{\partial \Pi^f}{\partial p_i} = -h_i'\Psi' - g_i' - (p_i - c_i)(h_i''\Psi' + g_i'') - \sum_{j \in f}(p_j - c_j)h_j'h_i'\Psi''.$$

Therefore, at any optimum,

$$\frac{p_i - c_i}{p_i}\iota_i(p_i) - \frac{g_i'(p_i) + (p_i - c_i)g_i''(p_i)}{h_i'(p_i)\Psi'(H)} = \underbrace{1 + \frac{\Psi''(H)}{\Psi'(H)}\sum_{j \in f}(p_j - c_j)h_j'(p_i)}_{\equiv \mu^f}.$$

Note that the left-hand side of the above condition only depends on p_i and H, whereas the right-hand side, which we call μ^f, is independent of the identity of product i. Therefore, for a given aggregator level H, firm f's optimal strategy can still be summarized by uni-dimensional sufficient statistic μ^f. Note that the corresponding pricing function r_i now depends on H and μ^f, which complicates the analysis.

V Proof of Proposition 7

Proof. It is straightforward to show, using standard differential equation techniques, that $-x\frac{h''(x)}{h'(x)} = \widetilde{\iota}(x)$ for all x if and only if $h = h^{\alpha,\beta}$ for some $\alpha \neq 0$ and $\beta \in \mathbb{R}$. All we need to do now is look for the set of pairs (α, β) such that $h^{\alpha,\beta} \in \mathcal{H}^\iota$.

Note that, for all α, β,

$$h^{\alpha,\beta\prime}(x) = -\alpha \exp\left(-\int_1^x \frac{\iota(u)}{u} du\right),$$

i.e., $h^{\alpha,\beta\prime}$ has the same sign as $-\alpha$. It follows that $h^{\alpha,\beta}$ cannot be in $\mathcal{H}^\iota$ if $\alpha \leq 0$. In addition, if $h^{\alpha,\beta} \in \mathcal{H}^\iota$ for some $\alpha > 0$ and $\beta \in \mathbb{R}$, then $h^{\alpha',\beta} \in \mathcal{H}^\iota$ for all $\alpha' > 0$. Therefore, we can set α equal to 1 without loss of generality.

The problem now boils down to finding the set of β's such that $h^\beta \equiv h^{1,\beta}$ is strictly positive, decreasing and log-convex. We already know that $h^{\beta\prime} < 0$. Therefore, the fact that h^β has to be decreasing does not impose any constraint on β.

Next, we show that $\lim_\infty h^0$ (which exists, since h^0 is monotone) is finite and strictly negative. It is trivial to see that this limit is strictly negative. Let $x^0 > 0$ such that $\widetilde{\iota}(x^0) > 1$. Proving that $\lim_\infty h^0$ is finite is equivalent to showing that function $t \mapsto \exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u} du\right)$ is integrable on $[x^0, \infty)$. For every $t \geq x^0$,

$$\begin{aligned}
\exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u} du\right) &\leq \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u} du - \int_{x^0}^t \frac{\iota(x^0)}{u} du\right), \\
&= \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u} du\right) \exp\left(-\widetilde{\iota}(x^0) \log\left(\frac{t}{x^0}\right)\right), \qquad \text{(xiv)} \\
&= \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u} du\right) \left(\frac{t}{x^0}\right)^{-\widetilde{\iota}(x^0)}.
\end{aligned}$$

The last expression is integrable on $[x^0, \infty)$, since $\widetilde{\iota}(x^0) > 1$. Therefore, $t \mapsto \exp\left(-\int_1^t \frac{\widetilde{\iota}(u)}{u} du\right)$ is integrable on $[x^0, \infty)$ and $\hat{\beta} \equiv \lim_\infty h^0$ is finite and strictly negative. It follows that function h^β is strictly positive if and only if $\beta \geq \hat{\beta}$.

Let $\beta \geq \hat{\beta}$. Then,

$$\frac{d}{dx}\frac{h^{\beta\prime}(x)}{h^\beta(x)} = \frac{h^{\beta\prime\prime}(x)h^\beta(x) - \left(h^{\beta\prime}(x)\right)^2}{h^\beta(x)^2} = \frac{1}{x}\frac{-h^{\beta\prime}(x)}{h^\beta(x)}\left(\widetilde{\iota}(x) - x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}\right).$$

Therefore, h^β is log-convex if and only if $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^\beta(x)}$ for all $x > 0$. Since $h^\beta(x)$ increases with β and $h^{\beta\prime}(x)$ does not depend on β, it follows that, if h^{β} is log-convex and $\beta' > \beta$, then $h^{\beta'}$ is also log-convex.

Moreover, using (xiv), we see that, for every $x > x^0$,

$$
\begin{aligned}
-xh^{\beta\prime}(x) &\leq x \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u} du\right) \left(\frac{x}{x^0}\right)^{-\widetilde{\iota}(x^0)}, \\
&= \exp\left(-\int_1^{x^0} \frac{\widetilde{\iota}(u)}{u} du\right) \left(x^0\right)^{\widetilde{\iota}(x^0)} x^{1-\widetilde{\iota}(x^0)} \underset{x\to\infty}{\longrightarrow} 0,
\end{aligned}
$$

where the last line follows from the fact that $\widetilde{\iota}(x^0) > 1$.

Let $\beta > \hat{\beta}$. Then, $\lim_{\infty} h^{\beta} > 0$, and therefore, $\lim_{x\to\infty} x\frac{-h^{\beta\prime}(x)}{h^{\beta}(x)} = 0$. Since $\lim_{\infty}\widetilde{\iota} > 0$, it follows that there exists $\hat{x}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^{\beta}(x)}$ whenever $x \geq \hat{x}$. In addition, since h^{β} increases with β, we also have that, for all $\beta' \geq \beta$, $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'\prime}(x)}{h^{\beta'}(x)}$ whenever $x \geq \hat{x}$.

Next, we turn our attention to $\lim_{x\to 0^+} \frac{-xh^{\beta\prime}(x)}{h^{\beta}(x)}$. Note that

$$
\frac{d}{dx}(-xh^{\beta\prime}(x)) = -h^{\beta\prime}(x)\left(1-\widetilde{\iota}(x)\right).
$$

Therefore, if $\lim_{0^+}\widetilde{\iota} > 1$ or $\lim_{0^+}\widetilde{\iota} < 1$, then $x \mapsto (-xh^{\beta\prime}(x))$ is monotone in the neighborhood of zero, and $\lim_{x\to 0^+} -xh^{\beta\prime}(x)$ exists. If instead $\lim_{0^+}\widetilde{\iota} = 1$, then, by monotonicity, either there exists $\varepsilon > 0$ such that $\widetilde{\iota}(x) = 1$ for all $x \in (0,\varepsilon)$, or $\widetilde{\iota}(x) > 1$ for all $x > 0$. In both cases, $x \mapsto (-xh^{\beta\prime}(x))$ is still monotone in the neighborhood of zero, and $\lim_{x\to 0^+} -xh^{\beta\prime}(x)$ therefore exists. Note that $\lim_{0^+} h^{\beta}$ trivially exists, since h^{β} is monotone.

We distinguish two cases. Suppose first that $\lim_{x\to 0^+} -xh^{\beta\prime}(x)$ is finite, and denote this limit by l. If $\lim_{0^+} h^{\beta} = \infty$, then

$$
\widetilde{\iota}(x) - x\frac{-h^{\beta\prime}(x)}{h^{\beta}(x)} \underset{x\to 0^+}{\longrightarrow} \lim_{0^+}\widetilde{\iota} > 0.
$$

Therefore, there exists $\tilde{x} > 0$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^{\beta}(x)}$ for all $x \in (0,\tilde{x}]$. In addition, the inequality also holds if we replace β by $\beta' \geq \beta$. If, instead, $\lim_{0^+} h^{\beta} < \infty$, then

$$
\widetilde{\iota}(x) - x\frac{-h^{\beta\prime}(x)}{h^{\beta}(x)} \underset{x\to 0^+}{\longrightarrow} \underbrace{\lim_{0^+}\widetilde{\iota}}_{>0} - \frac{l}{\lim_{0^+} h^{\hat{\beta}} + \beta - \hat{\beta}},
$$

which is strictly positive for β high enough. For such a high enough β, we again obtain the existence of an $\tilde{x}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^{\beta}(x)}$ for all $x \in (0,\tilde{x}]$.

Next, assume instead that $\lim_{x\to 0^+} -xh^{\beta\prime}(x) = \infty$. Let $M > 0$. There exists $\varepsilon > 0$ such that $h^{\beta\prime}(x) < -M/x$ whenever $x \leq \varepsilon$. Integrating this inequality between x and ε, we see that

$$h^{\beta}(x) > h^{\beta}(\varepsilon) + M \log \frac{\varepsilon}{x} \underset{x\to 0^+}{\longrightarrow} \infty.$$

Therefore, $\lim_{0^+} h^{\beta} = \infty$, and we can apply l'Hospital's rule:

$$\lim_{x\to 0^+} \frac{-xh^{\beta\prime}(x)}{h^{\beta}(x)} = \lim_{x\to 0^+} \frac{-xh^{\beta\prime\prime}(x) - h^{\beta\prime}(x)}{h^{\beta\prime}(x)} = \lim_{0^+} \widetilde{\iota} - 1.$$

Therefore,

$$\widetilde{\iota}(x) - x\frac{-h^{\beta\prime}(x)}{h^{\beta}(x)} \underset{x\to 0^+}{\longrightarrow} 1 > 0.$$

Again, this gives us the existence of an $\tilde{x}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^{\beta}(x)}$ for all $x \in (0, \tilde{x}]$.

To summarize, we have found a $\beta > \hat{\beta}$ and two strictly positive reals $\tilde{x}$ and $\hat{x}$ such that for all $\beta' \geq \beta$, $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'\prime}(x)}{h^{\beta'}(x)}$ whenever $x \geq \hat{x}$ or $x \leq \tilde{x}$. If $\tilde{x} \geq \hat{x}$, then we are done: there exists $\beta > \hat{\beta}$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta\prime}(x)}{h^{\beta}(x)}$ for all $x > 0$. Assume instead that $\tilde{x} < \hat{x}$. Then, for every $\beta' \geq \beta$ and $x \in [\tilde{x}, \hat{x}]$,

$$\begin{aligned} x\frac{-h^{\beta'\prime}(x)}{h^{\beta'}(x)} &\leq x\frac{-h^{\beta'\prime}(x)}{h^{\beta'}(\hat{x})}, \text{ since } h^{\beta'} \text{ is non-increasing,} \\ &= x\frac{-h^{\beta\prime}(x)}{h^{\beta}(\hat{x}) + \beta' - \beta}, \text{ since } h^{\beta'} - h^{\beta} = \beta' - \beta, \\ &\leq \underbrace{\max_{t\in[\tilde{x},\hat{x}]} \left(-th^{\beta\prime}(t)\right)}_{\text{finite, by continuity and compactness}} \frac{1}{h^{\beta}(\hat{x}) + \beta' - \beta} \underset{\beta'\to\infty}{\longrightarrow} 0. \end{aligned}$$

Therefore, there exists $\beta' \geq \beta$ such that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'\prime}(x)}{h^{\beta'}(x)}$ for all $x \in [\tilde{x}, \hat{x}]$. It follows that $\widetilde{\iota}(x) \geq x\frac{-h^{\beta'\prime}(x)}{h^{\beta'}(x)}$ for all $x > 0$.

This implies that set

$$B \equiv \left\{\beta \geq \hat{\beta} : \quad h^{\beta} \text{ is log-convex}\right\}$$

is non-empty. In addition, we also know that if $\beta' > \beta$ and $\beta \in B$, then $\beta' \in B$. Put $\underline{\beta} = \inf B$. Assume for a contradiction that $\underline{\beta} \notin B$. Then, there exists $x > 0$ such that

$$\widetilde{\iota}(x) < x\frac{-h^{\underline{\beta}\prime}(x)}{h^{\underline{\beta}}(x)}.$$

Then, by continuity of h^{β} in β, there exists $\beta' > \underline{\beta}$ such that

$$\widetilde{\iota}(x) < x\frac{-h^{\beta'\prime}(x)}{h^{\beta'}(x)}.$$

But then, $\beta' \in B$ and $h^{\beta'}$ is not log-convex, a contradiction. Therefore, the set of β's such that h^β is positive, decreasing and log-convex is $[\underline{\beta}, \infty)$. $\square$

VI Quantity Competition

VI.1 The Demand System

We work with the following family of inverse demand systems:

$$P_i(x) = \frac{h_i'(x_i)}{\sum_{j \in \mathcal{N}} h_j(x_j)},$$

where x_j is the output of good j. We assume that $h_i > 0$ and $h_i' > 0$, i.e., products are substitutes. We also assume that $h_i'' < 0$, which ensures that, under monopolistic competition, the inverse demand for product i is strictly decreasing everywhere. This also implies $\partial P_i / \partial x_i < 0$.

The direct utility function associated with this demand system is $U(x) = \log \sum_{j \in \mathcal{N}} h_j(x_j)$. We claim that U is strictly concave. To see this, assume without loss of generality that $\mathcal{N} = \{1, \dots, N\}$, and note that the Jacobian of inverse demand system P is given by:

$$J = \frac{1}{H^2} \begin{pmatrix} h_1''H - (h_1')^2 & -h_1'h_2' & \dots & -h_1'h_n' \\ -h_2'h_1' & h_2''H - (h_2')^2 & \dots & -h_2'h_n' \\ \vdots & \vdots & \ddots & \vdots \\ -h_n'h_1' & -h_n'h_2' & \dots & h_n''H - (h_n')^2 \end{pmatrix},$$

where $H = \sum_{j \in \mathcal{N}} h_j(x_j)$. Define $\gamma_i \equiv \frac{(h_i')^2}{h_i''} (< 0)$. By Lemma II, J is negative definite if and only if matrix

$$\begin{pmatrix} 1 - \frac{H}{\gamma_1} & 1 & \dots & 1 \\ 1 & 1 - \frac{H}{\gamma_2} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 - \frac{H}{\gamma_n} \end{pmatrix} = \mathcal{M}\left(\left(\frac{H}{\gamma_i} \right)_{1 \le i \le n} \right)$$

is positive definite. By Lemma I, the characteristic polynomial of this matrix is:

$$\mathcal{P}(X) = (-1)^n \left(\prod_{i=1}^n \left(\frac{H}{\gamma_i} + X \right) - \sum_{j=1}^n \prod_{j \neq i} \left(\frac{H}{\gamma_i} + X \right) \right).$$

For every $X \leq 0$,

$$\mathcal{P}(X) = \underbrace{(-1)^n \prod_{i=1}^{n} \left(\frac{H}{\gamma_i} + X\right)}_{>0} \underbrace{\left(1 - \sum_{j=1}^{n} \frac{1}{\underbrace{\frac{H}{\gamma_j} + X}_{<0}}\right)}_{>0} > 0.$$

Therefore, $\mathcal{P}$ does not have non-positive roots, and U is strictly concave.

VI.2 Assumptions and Technical Preliminaries

We make two assumptions on the limits of h_i'. First, we assume that $\lim_0 h_i' = \infty$. This means that, under monopolistic competition, a firm can always make strictly positive profits by supplying a strictly positive quantity. Second, we assume that $\lim_\infty h_i' = 0$. In other words, the monopolistic competition price of good i goes to 0 as x_i tends to infinity.

Moreover, we assume that monopolistic competition inverse demand functions satisfy Marshall's second law of demand: $|\iota_i|$ is non-decreasing for every i, where $\iota_i(x_i) = x_i \frac{h_i''(x_i)}{h_i'(x_i)}$. Since $h_i' > 0$ and $h_i'' < 0$, this means that $\iota_i' \leq 0$.

Next, we use these assumptions to establish a few basic facts about functions h_i and ι_i. Note first that $\lim_{x_i \to 0} x_i h_i'(x_i) = 0$. To see this, note that, by the fundamental theorem of calculus,

$$h_i(x_i) - h_i(0) = \int_0^{x_i} h_i'(t)dt \geq x_i h_i'(x_i) \geq 0,$$

where the first inequality follows from the fact that $h_i'' < 0$. By the sandwich theorem, it follows that $\lim_{x_i \to 0} x_i h_i'(x_i) = 0$.

Next, let $\bar{\mu}_i = 1 + \lim_0 \iota_i$. Since $\iota_i \leq 0$ and ι_i is monotone, $\bar{\mu}_i$ exists, and $\bar{\mu}_i \leq 1$. Assume for a contradiction that $\bar{\mu}_i \leq 0$. Then, since ι_i is non-increasing, it follows that $\iota_i(x_i) \leq -1$ for every x_i. Therefore,

$$\frac{d}{dx_i} \left(x_i h_i'(x_i)\right) = x_i h_i''(x_i) + h_i'(x_i) \leq 0.$$

Since $\lim_{x_i \to 0} x_i h_i'(x_i) = 0$, it follows that $x_i h_i'(x_i) \leq 0$ for every x_i. Therefore, $h_i' \leq 0$, a contradiction. We conclude that $\bar{\mu}_i \in (0, 1]$ for every i.

VI.3 The Quantity-Setting Game and the Firm's Profit-Maximization Problem

A quantity-setting game is a triple $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$, where $(h_j)_{j \in \mathcal{N}}$ is an inverse demand system, $\mathcal{F}$ is a partition of the set of products, and $(c_j)_{j \in \mathcal{N}}$ is a vector of marginal costs. The profit of firm $f \in \mathcal{F}$ can be written as follows:

$$\Pi^f(x) = \sum_{\substack{j \in f \\ x_j > 0}} \left(\frac{h_j'(x_j)}{\sum_{i \in \mathcal{N}} h_i(x_i)} - c_j \right) x_j.$$

Fix a firm $f \in \mathcal{F}$, and let $(x_j)_{j \in \mathcal{N} \setminus f}$ such that $\sum_{j \in \mathcal{N} \setminus f} h_j(x_j) > 0$. Then, we claim that maximization problem

$$\max_{(x_j)_{j \in f} \in [0,\infty)^f} \Pi^f \left((x_j)_{j \in f}, (x_j)_{j \in \mathcal{N} \setminus f} \right) \tag{xv}$$

has a solution. To see this, note that the assumptions made and the preliminary results derived in Section VI.2 imply that $\Pi^f(\cdot, (x_j)_{j \in \mathcal{N} \setminus f})$ is continuous on $[0, \infty)^f$. Moreover, since products are substitutes and $\lim_\infty h_i'(x_i) = 0$ for every i, there exists $M > 0$ such that for every $(x_j)_{j \in f} \in [0, \infty)^f$, there exists $(x_j')_{j \in f} \in [0, M]^f$ such that

$$\Pi^f \left((x_j)_{j \in f}, (x_j)_{j \in \mathcal{N} \setminus f} \right) < \Pi^f \left((x_j')_{j \in f}, (x_j)_{j \in \mathcal{N} \setminus f} \right).$$

Therefore, the sets of solutions of maximization problems (xv) and

$$\max_{(x_j)_{j \in f} \in [0,M]^f} \Pi^f \left((x_j)_{j \in f}, (x_j)_{j \in \mathcal{N} \setminus f} \right) \tag{xvi}$$

coincide. Since $\Pi^f(\cdot, (x_j)_{j \in \mathcal{N} \setminus f})$ is continuous and $[0, M]^f$ is compact, maximization problem (xvi) has a solution.

VI.4 The Additive Constant ι-Markup Property

We start by deriving first-order conditions under the assumption that all products are active. The derivative of firm f's profit with respect to x_k ($k \in f$) is given by:

$$\begin{aligned} \frac{\partial \pi^f}{\partial x_k} &= \frac{h_k'}{H} \left(-\frac{\sum_{j \in f} x_j h_j'}{H} + x_k \frac{h_k''}{h_k'} + \frac{\frac{h_k'}{H} - c_k}{\frac{h_k'}{H}} \right), \\ &= \frac{h_k'}{H} \left(-\frac{\sum_{j \in f} x_j h_j'}{H} + \iota_k + \frac{P_k - c_k}{P_k} \right), \end{aligned}$$

Therefore, if the first-order conditions hold at output vector $(x_k)_{k \in f}$, then, for every $k \in f$,

$$\frac{P_k - c_k}{P_k} + \iota_k = \frac{\sum_{j \in f} x_j h_j'}{H}.$$

Since the right-hand side of the above condition does not depend on k, it follows that an additive form of the constant ι-markup property holds:

$$\frac{P_k - c_k}{P_k} + \iota_k = \frac{P_l - c_l}{P_l} + \iota_l \equiv \mu^f, \quad \forall k, l \in f.$$

Under monopolistic competition, we would have $\mu^f = \frac{P_k - c_k}{P_k} + \iota_k = 0$. Under oligopoly, the firm internalizes its impact on the aggregator, and sets $\mu^f > 0$.

VI.5 Definition and Properties of Output Functions

Fix $H > 0$, and consider the following function:

$$\nu_k(x_k, H) = 1 - c_k \frac{H}{h_k'(x_k)} + \iota_k(x_k) \left(= \frac{P_k - c_k}{P_k} + \iota_k(x_k) \right).$$

ν_k maps an output level and an aggregator level into a ι-markup. Note that, contrary to the price-competition case, ν_k depends on H.

ν_k is differentiable, $\partial \nu_k / \partial x_k < 0$ (due to $h_k'' < 0$ and to Marshall's second law of demand), and $\partial \nu_k / \partial H < 0$. By the inverse function theorem, inverse function $\chi_k(\mu^f, H)$ is well-defined and differentiable, and satisfies $\partial \chi_k / \partial \mu^f < 0$ and $\partial \chi_k / \partial H < 0$. Output function χ_k maps a ι-markup and an aggregator level into an output level. It plays the same role as pricing function r_k in the paper.

For every $x_k > 0$,

$$\nu_k(x_k, H) < \sup_{\tilde{x}_k > 0} \nu_k(\tilde{x}_k, H) = \bar{\mu}_k.$$

Therefore, if $\mu^f \geq \bar{\mu}_k$, then ι-markup μ^f is not consistent with product k being sold. We therefore extend χ_k by continuity: $\chi_k(\mu^f, H) = 0$ whenever $\mu^f \geq \bar{\mu}_k$. Denote $\bar{\mu}^f = \max_{j \in f} \bar{\mu}_j$.

VI.6 Definition and Properties of Markup Fitting-In Functions

Next, we use the output functions defined in the previous subsection to reduce firm f's first-order conditions to a uni-dimensional equation:[3]

$$\mu^f = \frac{1}{H} \sum_{j \in f} \chi_j(\mu^f, H) h_j' \left(\chi_j(\mu^f, H) \right). \tag{xvii}$$

Since the right-hand side of condition (xvii) is strictly positive, we can safely restrict attention to strictly positive μ^fs. Note that, for every $k \in f$ and $\mu^f \in [0, \bar{\mu}_k)$,

$$\iota_k \left(\chi_k(\mu^f, H) \right) = \mu^f + c_k \frac{H}{h_k' \left(\chi_k(\mu^f, H) \right)} - 1 > -1.$$

Therefore, by definition of ι_k,

$$\chi_k(\mu^f, H) h_k'' \left(\chi_k(\mu^f, H) \right) + h_k' \left(\chi_k(\mu^f, H) \right) > 0.$$

Combining the above inequality with the fact that $\partial \chi_k / \partial \mu^f < 0$ for every k such that

[3] If the j-th term of the sum is such that $\bar{\mu}_j \leq \mu^f$, then $\chi_j(\mu^f, H) h_j'(\chi_j(\mu^f, H)) = \lim_{x_j \to 0} x_j h_j'(x_j) = 0$.

$\bar{\mu}_k > \mu^f$, it follows that the right-hand side of condition (xvii) is strictly decreasing in μ^f on interval $(0, \bar{\mu}^f)$, and identically equal to zero on interval $[\bar{\mu}^f, \infty)$. Since the left-hand side is strictly increasing in μ^f, there exists at most one μ^f such that firm f's simplified optimality condition holds.

If $\mu^f \geq \bar{\mu}^f \equiv \max_{k\in f} \bar{\mu}_k$, then the right-hand side of equation (xvii) is equal to zero while the left-hand side is strictly positive. If μ^f is equal to zero, then the right-hand side of equation (xvii) is strictly positive, and the left-hand side is equal to zero. Therefore, equation (xvii) has a unique solution, which we denote by $m^f(H)$. m^f is firm f's markup fitting-in function.

Totally differentiating equation (xvii) yields:[4]

$$d\mu^f = -\frac{dH}{H}\mu^f + \frac{1}{H}\sum_{j\in f}\left(\left.\frac{d(x_j h_j'(x_j))}{dx_j}\right|_{x_j=\chi_j} \left(\frac{\partial \chi_j}{\partial \mu^f}d\mu^f + \frac{\partial \chi_j}{\partial H}dH\right)\right).$$

Therefore,

$$m^{f\prime}(H) = \frac{\frac{-m^f}{H} + \frac{1}{H}\sum_{j\in f}\left(\left.\frac{d(x_j h_j'(x_j))}{dx_j}\right|_{x_j=\chi_j}\frac{\partial \chi_j}{\partial H}\right)}{1 - \frac{1}{H}\sum_{j\in f}\left(\left.\frac{d(x_j h_j'(x_j))}{dx_j}\right|_{x_j=\chi_j}\frac{\partial \chi_j}{\partial \mu^f}\right)},$$

which is strictly negative, since $\partial \chi_j/\partial \mu^f < 0$ and $\partial \chi_j/\partial H < 0$ for every j.

By monotonicity, $\lim_0 m^f$ and $\lim_\infty m^f$ exist. We will compute these limits in the next subsection.

VI.7 Definition and Properties of Output Fitting-In Functions

For every $k \in f$, let $X_k(H) = \chi_k\left(m^f(H), H\right)$. Function $H \mapsto (X_k(H))_{k\in f}$ is firm f's output fitting-in function.

We first argue that $\lim_\infty X_k$ exists and is equal to zero for every k. Assume for a contradiction that this is not the case. There exists $k \in f$, $(H^n)_{n\geq 0}$ and $\varepsilon > 0$ such that $H^n \underset{n\to\infty}{\longrightarrow} \infty$ and $X_k(H^n) > \varepsilon$ for every n. By definition of m^f, we also have that

$$\begin{aligned} m^f(H^n) &= 1 - c_k\frac{H^n}{h_k'(X_k(H^n))} + \iota_k(X_k(H^n)), \\ &< 1 - c_k\frac{H^n}{h_k'(\varepsilon)}, \text{ since } X_k(H^n) > \varepsilon,\ h_k'' < 0, \text{ and } \iota_k \leq 0, \\ &\underset{n\to\infty}{\longrightarrow} -\infty. \end{aligned}$$

Therefore, $m^f(H^n)$ is strictly negative for n high enough, a contradiction. Therefore, $\lim_\infty X_k = 0$.

[4] To ease notation, we ignore the fact that the sum should only span those j's that satisfy $\chi_j > 0$.

Next, we argue that $\lim_\infty m^f = 0$. Condition (xvii) can be rewritten as follows:

$$m^f(H) = \frac{1}{H} \sum_{j \in f} X_j(H) h_j'\left(X_j(H)\right).$$

Since, for every f, $\lim_\infty X_j = 0$ and $\lim_{x_j \to 0} x_j h_j'(x_j) = 0$, it follows that $\lim_\infty m^f = 0$.

Next, assume for a contradiction that X_k does not go to zero as H goes to 0 for some k in f. There exists $\varepsilon > 0$ and $(H^n)_{n \geq 0}$ such that $H^n \underset{n\to\infty}{\longrightarrow} 0$ and $X_k(H^n) > \varepsilon$ for every n. Recall that function $x_k \mapsto x_k h_k'(x_k)$ is strictly increasing on the relevant domain (see Section VI.6). It follows that, for every n,

$$\begin{aligned} m^f(H^n) &= \frac{1}{H^n} \sum_{j \in f} X_j(H^n) h_j'\left(X_j(H^n)\right), \\ &\geq \frac{1}{H^n} X_k(H^n) h_k'\left(X_k(H^n)\right), \\ &\geq \frac{1}{H^n} \varepsilon h_k'(\varepsilon), \\ &\underset{n\to\infty}{\longrightarrow} \infty. \end{aligned}$$

Since m^f is always below unity, this is a contradiction. Therefore, $\lim_0 X_k = 0$.

It follows immediately that $\lim_0 m^f = \bar{\mu}^f$. As competition intensifies (H goes up), firm f decreases its ι-markup from $\bar{\mu}^f$ (the monopoly case) to 0 (the monopolistic competition limit), and the set of products offered by firm f expands.

By contrast, output fitting-in function X_k is non-monotonic in H: $X_k(0) = X_k(\infty) = 0$, and $X_k(H) > 0$ for H high enough (if $\bar{\mu}_k < \bar{\mu}^f$, then $X_k = 0$ for H sufficiently low).

VI.8 Definition and Properties of the Aggregate Fitting-In Function

The aggregate fitting-in function is defined as follows:

$$\Gamma(H) = \sum_{f \in \mathcal{F}} \sum_{j \in f} h_j\left(X_j(H)\right).$$

Since $\Gamma(0) = \Gamma(\infty) = \sum_{j \in \mathcal{N}} h_j(0)$ and $\Gamma(H) > \sum_{j \in \mathcal{N}} h_j(0)$ for every $H > 0$, Γ is non-monotone.

In the following, we first establish the existence of an $H^* > 0$ such that $\Gamma(H^*) = H^*$. If $\lim_0 h_k > 0$ for some $k \in \mathcal{N}$, then this is trivial: Since Γ is continuous, $\Gamma(0) > 0$, and $\Gamma(\infty) < \infty$, existence of a fixed point follows from the intermediate value theorem.

Next, assume that $h_j(0) = 0$ for every j. Note first that, by L'Hospital's rule, for every j,

$$\lim_{x\to 0} \frac{h_j(x)}{x h_j'(x)} = \lim_{x\to 0} \frac{h_j'(x)}{h_j'(x) + x h_j''(x)} = \lim_{x\to 0} \frac{1}{1+\iota_j(x)} = \frac{1}{\bar{\mu}_k}.$$

To simplify the exposition, assume that $\bar{\mu}^f = \bar{\mu}_k$ for every f and $k \in f$. The case where this assumption is violated can be dealt with as we do in the proof of Lemma I (essentially, by taking an H small enough such that all firms are only supplying their high $\bar{\mu}_k$ products). Take some $\varepsilon > 0$ such that $|\mathcal{F}|(1-\varepsilon) > 1$. There exists $\hat{H} > 0$ such that $\frac{h_j(X_j(H))}{X_j(H)h_j'(X_j(H))} \geq (1-\varepsilon)\frac{1}{\bar{\mu}^f}$ for every $H < \hat{H}$, $f \in \mathcal{F}$ and $j \in f$. Moreover, for every $H < \hat{H}$,

$$\begin{aligned}
\frac{\Gamma(H)}{H} &= \sum_{f\in\mathcal{F}} \sum_{j\in f} \frac{h_j(X_j(H))}{H}, \\
&= \sum_{f\in\mathcal{F}} \sum_{j\in f} \frac{X_j(H)h_j'(X_j(H))}{H} \frac{h_j(X_j(H))}{X_j(H)h_j'(X_j(H))}, \\
&\geq (1-\varepsilon) \sum_{f\in\mathcal{F}} \frac{1}{\bar{\mu}^f} \frac{1}{H} \sum_{j\in f} X_j(H)h_j'(X_j(H)) \\
&= (1-\varepsilon) \sum_{f\in\mathcal{F}} \frac{1}{\bar{\mu}^f} m^f(H), \text{ by condition (xvii)}, \\
&\underset{H\to 0}{\longrightarrow} (1-\varepsilon) \sum_{f\in\mathcal{F}} 1, \text{ since } \lim_0 m^f = \bar{\mu}^f, \\
&= |\mathcal{F}|(1-\varepsilon), \\
&> 1.
\end{aligned}$$

It follows that $\Gamma(H) > H$ in the neighborhood of zero. The fact that $\lim_\infty \Gamma = 0$ and the continuity of Γ give us the existence of a fixed point.

VI.9 Equilibrium Uniqueness and Sufficiency of First-Order Conditions

In the previous subsection, we established the existence of an aggregator level H^* such that $\Gamma(H^*) = H^*$. Since we have not shown that first-order conditions are sufficient for global optimality, we cannot conclude that H^* is an equilibrium aggregator level.

Suppose that the following condition holds:

$$\sum_{j\in f} \left(H X_j'(H) h_j'\left(X_j(H)\right) - h_j\left(X_j(H)\right) \right) < 0, \quad \forall f \in \mathcal{F},\ \forall H > 0. \tag{xviii}$$

Fix a firm $f \in \mathcal{F}$ and a profile of outputs for firm f's rivals $(x_j)_{j\in\mathcal{N}\setminus f}$ such that $H^0 =$

$\sum_{j\in\mathcal{N}\setminus f} h_j(x_j) > 0$. Define

$$\Omega^f(H, H^0) = \frac{1}{H}\left(H^0 + \sum_{j\in f} h_j(X_j(H))\right).$$

The first-order conditions associated with firm f's profit-maximization problem hold at output vector $(x_j)_{j\in f}$ if and only if there exists $H > 0$ such that $x_j = X_j(H)$ for every $j \in f$, and $\Omega^f(H, H^0) = 1$. Since $\Omega^f(0, H^0) = \infty$, $\Omega^f(\infty, H^0) = 0$, and $\Omega^f(\cdot, H^0)$ is continuous, there exists $H > 0$ such that $\Omega^f(H, H^0) = 1$. Moreover, for every $H > 0$,

$$\begin{aligned}
\frac{\partial \Omega^f}{\partial H} &= \frac{1}{H^2}\left(\sum_{j\in f} X_j'(H)h_j'(X_j(H))H - (H^0 + \sum_{j\in f} h_j(X_j(H)))\right), \\
&< \frac{1}{H^2}\sum_{j\in f}\left(HX_j'(H)h_j'(X_j(H)) - h_j(X_j(H))\right), \\
&< 0, \text{ by condition (xviii).}
\end{aligned}$$

Therefore, $\Omega^f(\cdot, H^0)$ is strictly decreasing, and there exists a unique $H > 0$ such that $\Omega^f(H, H^0) = 1$. This means that there exists a unique output profile $(\tilde{x}_j)_{j\in f}$ for firm f such that firm f's first-order conditions hold. In Section VI.3, we have shown that firm f's profit maximization problem has a solution $(\hat{x}_j)_{j\in f}$. By necessity, first-order conditions must hold at output profile $(\hat{x}_j)_{j\in f}$. By uniqueness, $(\tilde{x}_j)_{j\in f} = (\hat{x}_j)_{j\in f}$. Therefore, first-order conditions are necessary and sufficient for optimality.

This implies that H is an equilibrium aggregator level if and only if H is a fixed point of the aggregate fitting-in function. Since we have established existence of such a fixed point, it follows that the quantity-setting game has a Nash equilibrium.

In fact, under condition (xviii), we can even prove that the quantity-setting game has a unique equilibrium. To see this, define $\Omega(H) = \Gamma(H)/H$. Then,

$$\Omega'(H) = \frac{1}{H^2}\left(\sum_{f\in\mathcal{F}}\sum_{j\in f} HX_j'(H)h_j'(X_j(H)) - \sum_{f\in\mathcal{F}}\sum_{j\in f} h_j(X_j(H))\right),$$

which is strictly negative by condition (xviii). Therefore, the aggregate fitting-in function has a unique fixed point, and the quantity-setting game has a unique equilibrium.

VI.10 The CES Case

In the following, we show that condition (xviii) holds in the CES case. For every $j \in \mathcal{N}$, let $h_j(x_j) = a_j x_j^\alpha$, where $a_j > 0$ is a quality parameter, and $\alpha \in (0, 1)$. Clearly, h_j is strictly increasing and strictly concave, $\lim_0 h_j' = \infty$, and $\lim_\infty h_j' = 0$. Moreover, $\iota_j = \alpha - 1$.

Note that, for every firm f,

$$m^f(H) = \frac{1}{H} \sum_{j \in f} X_j(H) h_j'(X_j(H)) = \frac{\alpha}{H} \sum_{j \in f} h_j(X_j(H)).$$

Therefore,

$$m^{f\prime}(H) = \frac{\alpha}{H^2} \sum_{j \in f} \left(H X_j'(H) h_j'(X_j(H)) - h_j(X_j(H)) \right).$$

Since $m^{f\prime} < 0$, it follows that $\sum_{j \in f} \left(H X_j'(H) h_j'(X_j(H)) - h_j(X_j(H)) \right) < 0$, i.e., condition (xviii) holds. Therefore, multiproduct-firm quantity-setting games with CES demands have a unique equilibrium.

VII Equilibrium Uniqueness

In this section, we fix a pricing game $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ satisfying Assumption 1. Define $\underline{p}_k = \underline{p}(h_k)$ (see Lemma III-(b)) for every k. As in Section III.2, we make explicit the dependence on c_k of functions ν_k and r_k by writing $\nu_k(p_k, c_k)$ and $r_k(\mu^f, c_k)$.[5] It is straightforward to show that ν_k is decreasing in c_k, and that r_k is increasing in c_k.

VII.1 Preliminaries

In this section, we prove several technical lemmas, which will allow us to derive firm-level conditions for equilibrium uniqueness.

We first show that the assumption that ρ_j is non-decreasing on $(\underline{p}_j, \infty)$ is equivalent to the convexity of the reciprocal of demand for product j. Let $D_j : (p_j, H^0) \in (\underline{p}_j, \infty) \times \mathbb{R}_{++} \mapsto -h_j'(p_j)/(h_j(p_j) + H^0)$.

Lemma VII. *The following assertions are equivalent:*

(i) $p_j \in (\underline{p}_j, \infty) \mapsto 1/D_j(p_j, H^0)$ *is convex for every* H^0.

(ii) ρ_j *is non-decreasing on* $(\underline{p}_j, \infty)$.

Proof. Note that, for every $p_j > \underline{p}_j$ and $H^0 > 0$,

$$\begin{aligned}
\frac{\partial^2}{\partial p_j^2} \frac{1}{D_j(p_j, H^0)} &= - \left(\frac{h_j + H^0}{h_j'} \right)'', \\
&= - \left(\frac{\left(h_j'\right)^2 - h_j'' \left(h_j + H^0\right)}{\left(h_j'\right)^2} \right)', \\
&= \left(\rho_j + \frac{H^0}{\gamma_j} \right)',
\end{aligned}$$

[5] Recall that $\nu_k(p_k, c_k) = \frac{p_k - c_k}{p_k} \iota_k(p_k)$.

$$= \rho_j' - \frac{\gamma_j' H^0}{\gamma_j^2},$$

which, by Lemma III-(d), is non-negative for every $p_j > \underline{p}_j$ and H^0 if and only if $\rho_j'(p_j) \geq 0$ for all $p_j > \underline{p}_j$. □

Next, we introduce the following notation. For every $f \in \mathcal{F}$, for every $\mu^f \in (1, \bar{\mu}^f)$,

$$\omega^f = \frac{\mu^f - 1}{\mu^f},$$
$$\bar{\omega}^f = \lim_{\mu^f \to \bar{\mu}^f} \frac{\mu^f - 1}{\mu^f},$$

and for every $k \in \mathcal{N}$, for every $x > \underline{p}_k$,

$$\chi_k(x) = \frac{\iota_k(x) - 1}{\iota_k(x)}.$$

The following lemma is useful to understand our uniqueness conditions:

Lemma VIII. *For every $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$ and $k \in f$:*

- *For every x such that $\chi_k(x) > \omega^f$, $1 - \omega^f \theta_k(x) > 0$.*
- *In particular, for every $c_k > 0$, for every $x \in \left[r_k\left(\frac{1}{1-\omega^f}, c_k\right), \infty\right)$, $1 - \omega^f \theta_k(x) > 0$.*
- *In particular, for every $x > \underline{p}_k$, $\chi_k(x)\theta_k(x) \leq 1$.*

Proof. Let $f \in \mathcal{F}$, $k \in f$, $\omega^f \in \left(0, \bar{\omega}^f\right)$, and x such that $\chi_k(x) > \omega^f$. Put $\mu^f = \frac{1}{1-\omega^f}$. Then, $\iota_k(x) > \mu^f$. Therefore, there exists $c > 0$ such that $\nu_k(x, c) = \mu^f$. We know from Lemma D that

$$\frac{\partial r_k}{\partial \mu^f}(\mu^f, c) = \frac{\gamma_k(r_k(\mu^f, c_k))}{\mu^f\left(-\gamma_k'\left(r_k(\mu^f, c_k)\right)\right) - (\mu^f - 1)\left(-h_k'\left(r_k(\mu^f, c_k)\right)\right)},$$
$$= \frac{\gamma_k(x)}{-\gamma_k'(x)\mu^f} \frac{1}{1 - \omega^f \theta_k(x)} > 0.$$

In addition, by Lemma III-(d), $\gamma_k'(x) < 0$. Therefore, $1 - \omega^f \theta_k(x) > 0$. This establishes the first bullet point in the statement of the lemma.

Next, let $c > 0$ and $x \geq r_k(\mu^f, c)$. Then, since $\nu_k(., c)$ is increasing, $\nu_k(x, c) \geq \mu^f$. Since $c > 0$, it follows that $\iota_k(x) > \mu^f$, and that $\chi_k(x) > \omega^f$. It follows from the first part of the lemma that $1 - \omega^f \theta_k(x) > 0$.

Finally, let $x > \underline{p}_k$. Put $\omega^f = \chi_k(x)$. Then, for every y such that $\chi_k(y) > \omega^f$, $1-\omega^f\theta_k(y) > 0$. By monotonicity of χ_k, this implies that, for every $y > x$, $1 - \chi_k(x)\theta_k(y) > 0$. Therefore, by continuity of θ_k, $\chi_k(x)\theta_k(x) \leq 1$. □

We will need to differentiate fitting-in functions:[6]

Lemma IX. *For every* $f \in \mathcal{F}$ *and* $H > 0$ *such that* $m^f(H) \notin \{\bar{\mu}_j\}_{j\in f}$*,* m^f *is* $\mathcal{C}^1$ *in a neighborhood of* H*, and*

$$m^{f\prime}(H) = -\frac{1}{H}\frac{m^f(H)(m^f(H)-1)}{1+m^f(H)(m^f(H)-1)\frac{\sum_{k\in f,\ m^f(H)<\bar{\mu}_k} r_k'\big(m^f(H)\big)\big(-\gamma_k'\big(r_k\big(m^f(H)\big)\big)\big)}{\sum_{k\in f,\ m^f(H)<\bar{\mu}_k}\gamma_k\big(r_k\big(m^f(H)\big)\big)}} < 0. \quad \text{(xix)}$$

Proof. Recall that $m^f(H)$ is the unique solution of equation

$$\psi(\mu^f, H) \equiv \mu^f\left(1 - \frac{\sum_{j\in f}\gamma_j\left(r_j(\mu^f)\right)}{H}\right) = 1.$$

Let $H^0 > 0$ such that $m^f(H^0) \neq \bar{\mu}_k$ for all $k \in f$, and choose $\varepsilon > 0$ such that $\big(m^f(H^0)-\varepsilon, m^f(H^0)+\varepsilon\big) \cap \{\bar{\mu}_k\}_{k\in f} = \emptyset$. We introduce the following notation:

$$\hat{f} = \left\{k \in f : \quad m^f(H^0) < \bar{\mu}_k\right\}.$$

Note that if $\hat{f}$ were empty, then $\psi\left(m^f(H^0), H^0\right)$ would be equal to $m^f(H^0) > 1$, a contradiction. Define

$$\hat{\psi} :\ (\mu^f, H) \in \big(m^f(H^0)-\varepsilon, m^f(H^0)+\varepsilon\big) \times \mathbb{R}_{++} \mapsto \mu^f\left(1 - \frac{\sum_{j\in\hat{f}}\gamma_j\left(r_j(\mu^f)\right)}{H}\right),$$

and note that $\hat{\psi}(\mu^f, H) = \psi(\mu^f, H)$ for all $(\mu^f, H) \in \big(m^f(H^0)-\varepsilon, m^f(H^0)+\varepsilon\big) \times \mathbb{R}_{++}$. In addition, $\hat{\psi}\left(m^f(H^0), H^0\right) = 1$, $\hat{\psi}$ is $\mathcal{C}^1$,

$$\begin{aligned}\frac{\partial\hat{\psi}}{\partial\mu^f}\left(m^f(H^0), H^0\right) &= 1 - \frac{\sum_{k\in\hat{f}}\gamma_k}{H^0} + m^f(H^0)\left(-\frac{\sum_{k\in\hat{f}} r_k'\gamma_k'}{H^0}\right), \\ &= \frac{1}{m^f(H^0)} + (m^f(H^0)-1)\frac{\sum_{k\in\hat{f}} r_k'\left(-\gamma_k'\right)}{\sum_{k\in\hat{f}}\gamma_k},\end{aligned}$$

which is strictly positive, and

$$\frac{\partial\hat{\psi}}{\partial H}\left(m^f(H^0), H^0\right) = m^f(H^0)\frac{\sum_{k\in\hat{f}}\gamma_k}{(H^0)^2} = \frac{m^f(H^0)-1}{H^0}.$$

By the implicit function theorem, there exist $\eta > 0$ and a $\mathcal{C}^1$ function

$$\hat{m}^f : (H^0-\eta, H^0+\eta) \longrightarrow \big(m^f(H^0)-\varepsilon, m^f(H^0)+\varepsilon\big)$$

[6] In the statement and proof of this lemma, we drop argument c_k from function r_k to ease notation.

such that $\hat{\psi}\left(\hat{m}^f(H), H\right) = 1$ for all $H \in (H^0 - \eta, H^0 + \eta)$. In addition,

$$\hat{m}^{f\prime}(H^0) = -\frac{1}{H^0} \frac{m^f(H^0)\left(m^f(H^0) - 1\right)}{1 + m^f(H^0)\left(m^f(H^0) - 1\right) \frac{\sum_{k \in \hat{f}} r_k'\left(-\gamma_k'\right)}{\sum_{k \in \hat{f}} \gamma_k}},$$

which is indeed strictly negative. Since functions ψ and $\hat{\psi}$ coincide on $\left(m^f(H^0) - \varepsilon, m^f(H^0) + \varepsilon\right) \times \mathbb{R}_{++}$, and by uniqueness of m^f, it follows that m^f and $\hat{m}^f$ coincide on $(H^0 - \eta, H^0 + \eta)$. Therefore, m^f is $\mathcal{C}^1$ in an open neighborhood of H^0, and $m^{f\prime}(H^0) = \hat{m}^{f\prime}(H^0)$. □

Lemma X. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. If, for every $f \in \mathcal{F}$,*

$$\forall \omega^f \in (0, \bar{\omega}^f), \quad \left(\sum_{k \in f} \frac{\omega^f \theta_k}{1 - \omega^f \theta_k} \gamma_k\right)\left(\frac{1}{\sum_{k \in f} h_k} - \frac{\omega^f}{\sum_{k \in f} \gamma_k}\right) < 1, \qquad \text{(xx)}$$

where, for every k, functions θ_k, γ_k and h_k are all evaluated at point $p_k = r_k\left(\frac{1}{1-\omega^f}, c_k\right)$, then pricing game $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ has a unique equilibrium.

Proof. By Theorem 1, $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ has a pricing equilibrium. To prove that there is only one equilibrium, we show that $\Omega(H) = \Gamma(H)/H$ is strictly decreasing. Let $H > 0$, and, for every $f \in \mathcal{F}$, $\mu^f = m^f(H)$ and $\omega^f = \frac{\mu^f - 1}{\mu^f}$. Then,

$$\begin{aligned} H^2 \Omega'(H) &= H \sum_{f \in \mathcal{F}} m^{f\prime(H)} \sum_{k \in f} r_k'\left(\mu^f\right) h_k'\left(r_k\left(\mu^f\right)\right) - \sum_{f \in \mathcal{F}} \sum_{k \in \mathcal{N}} h_k\left(r_k\left(\mu^f\right)\right), \\ &= \sum_{f \in \mathcal{F}} \left(\frac{\mu^f(\mu^f - 1)}{1 + \mu^f(\mu^f - 1)\frac{\sum_{k \in f} r_k'\left(-\gamma_k'\right)}{\sum_{k \in f} \gamma_k}} \left(\sum_{k \in f} r_k'\left(-h_k'\right)\right) - \sum_{k \in f} h_k\right), \text{ by Lemma IX.} \end{aligned}$$

Therefore, a sufficient condition for this derivative to be strictly negative is that, for all $f \in \mathcal{F}$,

$$\frac{\mu^f(\mu^f - 1)\frac{\sum_{k \in f} r_k'\left(-h_k'\right)}{\sum_{k \in f} h_k}}{1 + \mu^f(\mu^f - 1)\frac{\sum_{k \in f} r_k'\left(-\gamma_k'\right)}{\sum_{k \in f} \gamma_k}} < 1. \qquad \text{(xxi)}$$

Let $f \in \mathcal{F}$. Then,

$$\begin{aligned} \text{(xxi)} &\iff (\mu^f - 1)\left(\frac{\sum_{k \in f} \mu^f r_k'\left(-h_k'\right)}{\sum_{k \in f} h_k} - \frac{\sum_{k \in f} \mu^f r_k'\left(-\gamma_k'\right)}{\sum_{k \in f} \gamma_k}\right) < 1, \\ &\iff (\mu^f - 1)\left(\frac{\sum_{k \in f} \mu^f \frac{\gamma_k\left(-h_k'\right)}{\mu^f\left(-\gamma_k'\right) - (\mu^f - 1)\left(-h_k'\right)}}{\sum_{k \in f} h_k} - \frac{\sum_{k \in f} \mu^f \frac{\gamma_k\left(-\gamma_k'\right)}{\mu^f\left(-\gamma_k'\right) - (\mu^f - 1)\left(-h_k'\right)}}{\sum_{k \in f} \gamma_k}\right) < 1, \end{aligned}$$

$$\iff (\mu^f - 1)\left(\frac{\sum_{k\in f}\frac{\theta_k}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f} h_k} - \frac{\sum_{k\in f}\frac{1}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f}\gamma_k}\right) < 1,$$

$$\iff (\mu^f - 1)\left(\frac{\sum_{k\in f}\frac{\theta_k}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f} h_k} - 1 - \frac{\sum_{k\in f}\frac{\omega^f\theta_k}{1-\omega^f\theta_k}\gamma_k}{\sum_{k\in f}\gamma_k}\right) < 1,$$

$$\iff (\mu^f - 1)\left(-1 + \sum_{k\in f}\frac{\theta_k}{1-\omega^f\theta_k}\gamma_k\left(\frac{1}{\sum_{k\in f} h_k} - \frac{\omega^f}{\sum_{k\in f}\gamma_k}\right)\right) < 1,$$

$$\iff \left(\sum_{k\in f}\frac{\omega^f\theta_k}{1-\omega^f\theta_k}\gamma_k\right)\left(\frac{1}{\sum_{k\in f} h_k} - \frac{\omega^f}{\sum_{k\in f}\gamma_k}\right) < 1,$$

where, for every $k \in f$, functions θ_k, γ_k and h_k are evaluated at point $p_k = r_k(\mu^f) = r_k\left(\frac{1}{1-\omega^f}, c_k\right)$. Since condition (xx) holds by assumption, Ω is strictly decreasing. Therefore, the pricing game has a unique equilibrium. □

Lemma XI. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $(\underline{c}_k)_{k\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$. If, for every $f \in \mathcal{F}$,*

$$\begin{aligned}&\forall \omega^f \in (0, \bar{\omega}^f),\ \forall (x_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\frac{1}{1-\omega^f}, \underline{c}_k\right), \infty\right),\\ &\left(\sum_{k\in f}\frac{\omega^f\theta_k(x_k)}{1-\omega^f\theta_k(x_k)}\gamma_k(x_k)\right)\left(\frac{1}{\sum_{k\in f} h_k(x_k)} - \frac{\omega^f}{\sum_{k\in f}\gamma_k(x_k)}\right) < 1,\end{aligned} \tag{xxii}$$

or, equivalently, if

$$\begin{aligned}&\forall \omega^f \in (0, \bar{\omega}^f),\ \forall (x_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\frac{1}{1-\omega^f}, \underline{c}_k\right), \infty\right),\\ &\sum_{i,j\in f}\gamma_i(x_i)\gamma_j(x_j)\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_j(x_j)}{1-\omega^f\theta_i(x_i)} - \rho_i(x_i)\right) < 0,\end{aligned} \tag{xxiii}$$

then pricing game $((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ has a unique equilibrium for every $(c_j)_{j\in\mathcal{N}} \in \prod_{j\in\mathcal{N}}[\underline{c}_j, \infty)$.

Proof. Assume that condition (xxii) holds, and let $(c_k)_{k\in\mathcal{N}} \in \prod_{k\in\mathcal{N}}[\underline{c}_k, \infty)$. We want to show that condition (xx) holds, so let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$ and $\mu^f = \frac{1}{1-\omega^f}$. Let $k \in f$ and $p_k = r_k(\mu^f, c_k)$. Since $c_k \geq \underline{c}_k$ and r_k is increasing in its second argument, it follows that $p_k \geq r_k(\mu^f, \underline{c}_k)$. Therefore, $(p_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\mu^f, \underline{c}_k\right), \infty\right)$. It follows that condition (xx) holds. By Lemma X, pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ has a unique equilibrium.

Finally, we show that conditions (xxii) and (xxiii) are equivalent. Let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$, and $(x_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\frac{1}{1-\omega^f}, \underline{c}_k\right), \infty\right)$. Then,

$$\begin{aligned}
&\left(\sum_{k\in f}\frac{\omega^f\theta_k}{1-\omega^f\theta_k}\gamma_k\right)\left(\frac{1}{\sum_{k\in f}h_k}-\frac{\omega^f}{\sum_{k\in f}\gamma_k}\right)<1\\
\iff&\left(\sum_{i\in f}\frac{\omega^f\theta_i}{1-\omega^f\theta_i}\gamma_i\right)\left(\sum_{j\in f}\left(\gamma_j-\omega^f h_j\right)\right)-\left(\sum_{i\in f}h_i\right)\left(\sum_{j\in f}\gamma_j\right)<0,\\
\iff&\left(\sum_{i\in f}\frac{\omega^f\theta_i}{1-\omega^f\theta_i}\gamma_i\right)\left(\sum_{j\in f}\gamma_j\left(1-\omega^f\rho_j\right)\right)-\left(\sum_{i\in f}\rho_i\gamma_i\right)\left(\sum_{j\in f}\gamma_j\right)<0,\\
\iff&\sum_{i,j\in f}\gamma_i\gamma_j\left(\omega^f\theta_i\frac{1-\omega^f\rho_j}{1-\omega^f\theta_i}-\rho_i\right)<0. \qquad \square
\end{aligned}$$

Lemma XII. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. If, for every $f \in \mathcal{F}$,*

$$\begin{aligned}
&\forall\omega^f\in(0,\bar{\omega}^f),\ \forall\,(x_k)_{k\in f}\in\left\{(x_k)_{k\in f}\in\mathbb{R}^f_{++}:\ \forall k\in f,\ \chi_k(x_k)>\omega^f\right\},\\
&\left(\sum_{k\in f}\frac{\omega^f\theta_k(x_k)}{1-\omega^f\theta_k(x_k)}\gamma_k(x_k)\right)\left(\frac{1}{\sum_{k\in f}h_k(x_k)}-\frac{\omega^f}{\sum_{k\in f}\gamma_k(x_k)}\right)<1,
\end{aligned}\tag{xxiv}$$

or, equivalently, if

$$\begin{aligned}
&\forall\omega^f\in(0,\bar{\omega}^f),\ \forall\,(x_k)_{k\in f}\in\left\{(x_k)_{k\in f}\in\mathbb{R}^f_{++}:\ \forall k\in f,\ \chi_k(x_k)>\omega^f\right\},\\
&\sum_{i,j\in f}\gamma_i(x_i)\gamma_j(x_j)\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_j(x_j)}{1-\omega^f\theta_i(x_i)}-\rho_i(x_i)\right)<0,
\end{aligned}\tag{xxv}$$

then pricing game $((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ has a unique equilibrium for every $(c_j)_{j\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$.

Proof. Let $(c_k)_{k\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$, and assume that condition (xxiv) holds. Let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$ and $\mu^f = 1/(1-\omega^f)$. Let $(x_k)_{k\in f} \in \prod_{k\in f}\left[r_k(\mu^f, c_k), \infty\right)$. Then, for every $k \in f$,

$$\iota_k(x_k) > \nu_k(x_k, c_k) = \mu^f.$$

Therefore,

$$(x_k)_{k\in f} \in \left\{(x_k)_{k\in f} \in \mathbb{R}^f_{++} :\ \forall k \in f,\ \chi_k(x_k) > \omega^f\right\},$$

and, by condition (xxiv), condition (xxii) holds for $(\underline{c}_k)_{k\in\mathcal{N}} = (c_k)_{k\in\mathcal{N}}$. By Lemma XI, pricing game $\left(\mathcal{N}, (h_k)_{k\in\mathcal{N}}, \mathcal{F}, (c_k)_{k\in\mathcal{N}}\right)$ has a unique equilibrium. In addition, as shown in the proof of Lemma XI, conditions (xxiv) and (xxv) are equivalent. $\square$

All we need to do to prove Theorem 2 is show that, for every $f \in \mathcal{F}$, each of conditions (a) and (b) in Theorem 2 implies condition (xxiv) (or, equivalently, condition (xxv)), and that condition (c) implies condition (xx).

VII.2 Sufficiency of condition (a)

We prove the following lemma:

Lemma XIII. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. If $\min_{j\in f} \inf_{p_j > \underline{p}_j} \rho_j(p_j) \geq \max_{j\in f} \sup_{p_j>\underline{p}_j} \theta_j(p_j)$, then condition* (xxiv) *holds for firm f.*

Proof. We show that condition (xxv) holds for firm f. Let $f \in \mathcal{F}$, $\omega^f \in (0, \bar{\omega}^f)$, and

$$(x_k)_{k\in f} \in \left\{(x_k)_{k\in f} \in \mathbb{R}^f_{++} : \ \forall k \in f, \ \chi_k(x_k) > \omega^f\right\}.$$

Since for every $k \in f$, $\chi_k(x_k) > \omega^f$, it follows that $\iota_k(x_k) > 1$. Therefore, $x_k > \underline{p}_k$ for every k, and

$$\max_{k\in f} \theta_k(x_k) \leq \min_{k\in f} \rho_k(x_k).$$

Therefore,

$$\sum_{i,j\in f} \gamma_i\gamma_j \left(\omega^f\theta_i \frac{1-\omega^f\rho_j}{1-\omega^f\theta_i} - \rho_i\right) \leq \sum_{i,j\in f} \gamma_i\gamma_j \left(\omega^f\rho_i - \rho_i\right) = (\omega^f - 1)\sum_{i,j\in f}\gamma_i\gamma_j\rho_i < 0,$$

where the first inequality follows by Lemma VIII and $\max_{k\in f}\theta_k(x_k) \leq \min_{k\in f}\rho_k(x_k)$. Therefore, condition (xxv) holds for firm f. □

VII.3 Sufficiency of condition (b)

The aim of this section is to prove the following lemma:

Lemma XIV. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Suppose that, $\bar{\mu}^f \leq \mu^*(\simeq 2.78)$, and for every $j \in f$, $\lim_\infty h_j = 0$ and ρ_j is non-decreasing on $(\underline{p}_j, \infty)$. Then, condition* (xxiv) *holds for firm f.*

This lemma is proven in several steps. Start with the following technical lemmas:

Lemma XV. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Suppose that $\bar{\mu}^f < \infty$, and for every $j \in f$, $\lim_\infty h_j = 0$ and ρ_j is non-decreasing on $(\underline{p}_j, \infty)$. Then, for every $\omega^f \in (0, \bar{\omega}^f)$, for every $k \in f$, for every $x > 0$ such that $\chi_k(x) > \omega^f$,*

$$\frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f} \leq \rho_k(x) \leq \frac{1}{\bar{\omega}^f}.$$

Proof. Let $k \in f$ and $\omega^f \in (0, \bar{\omega}^f)$. By Lemma III-(f), $\lim_\infty \rho_k = \frac{\bar{\mu}^f}{\bar{\mu}^f - 1} = \frac{1}{\bar{\omega}^f}$. In addition, ρ_k is non-decreasing. Therefore, $\rho_k(x) \leq \frac{1}{\bar{\omega}^f}$ for all $x > \underline{p}_k$. In particular, this inequality is also satisfied if x is such that $\chi_k(x) > \omega^f$.

In addition, $\rho_k(x) = \iota_k(x)\frac{h_k(x)}{-xh'_k(x)}$. Therefore,

$$\begin{aligned}
\frac{d\log\rho_k(x)}{dx} &= \frac{\iota'_k(x)}{\iota_k(x)} + \left(\frac{h'_k(x)}{h_k(x)} - \frac{1}{x} + \frac{h''_k(x)}{-h'_k(x)}\right), \\
&= \frac{\iota'_k(x)}{\iota_k(x)} + \frac{1}{x}\left(-\frac{\iota_k(x)}{\rho_k(x)} - 1 + \iota_k(x)\right), \\
&= \frac{\iota'_k(x)}{\iota_k(x)} + \frac{\iota_k(x)}{x\rho_k(x)}\left(\rho_k(x)\chi_k(x) - 1\right), \\
&\leq \frac{\iota'_k(x)}{\iota_k(x)},
\end{aligned}$$

where the last inequality follows from the fact that $\chi_k(x) \leq \bar{\omega}^f$ and $\rho_k(x) \leq \frac{1}{\bar{\omega}^f}$. Therefore, for all $x > \underline{p}_k$,

$$\log\left(\frac{1}{\bar{\omega}^f\rho_k(x)}\right) = \int_x^\infty \frac{\rho'_k(t)}{\rho_k(t)}dt \leq \int_x^\infty \frac{\iota'_k(t)}{\iota_k(t)}dt = \log\left(\frac{\bar{\mu}^f}{\iota_k(x)}\right) = \log\left(\frac{1-\chi_k(x)}{1-\bar{\omega}^f}\right).$$

Therefore,

$$\rho_k(x) \geq \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\chi_k(x)},\ \forall x > \underline{p}_k.$$

In particular, if $\chi_k(x) > \omega^f$, then

$$\rho_k(x) \geq \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}.$$

□

Lemma XVI. *For every $\bar{\omega} \in (0,1]$, for every $\omega \in (0,\bar{\omega})$, define*

$$\phi_{\omega,\bar{\omega}} : (y,z) \in \left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2 \mapsto \omega y\frac{1-\omega z}{1-\omega y} + \omega z\frac{1-\omega y}{1-\omega z} - y - z.$$

There exists a threshold $\omega^* \in (0,1)$ ($\omega^* \simeq 0.64$) such that if $\bar{\omega} \leq \omega^*$, then $\phi_{\omega,\bar{\omega}} \leq 0$ for all $\omega \in (0,\bar{\omega})$.

Proof. Let $\bar{\omega} \in (0,1)$ and $\omega \in (0,\bar{\omega})$. Define

$$M(\omega,\bar{\omega}) = \max_{(y,z)\in\left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2} \phi_{\omega,\bar{\omega}}(y,z).$$

Notice that $\phi_{\omega,\bar{\omega}}(y,z) = \phi_{\omega,\bar{\omega}}(z,y)$ for every y and z. It follows that

$$M(\omega,\bar{\omega}) = \max_{\substack{(y,z)\in\left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2 \\ y\leq z}} \phi_{\omega,\bar{\omega}}(y,z).$$

Let $\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega} \leq y \leq z \leq \frac{1}{\bar{\omega}}$. Then,

$$\begin{aligned}
\frac{\partial \phi_{\omega,\bar{\omega}}}{\partial y} &= \frac{\omega(1-\omega z)}{(1-\omega y)^2} - \frac{\omega^2 z}{1-\omega z} - 1, \\
&= \frac{1}{1-\omega z}\left(\omega\left(\frac{1-\omega z}{1-\omega y}\right)^2 - \omega^2 z - (1-\omega z)\right), \\
&\leq \frac{1}{1-\omega z}\left(\omega - \omega^2 z - (1-\omega z)\right), \text{ since } y \leq z, \\
&= \omega - 1 < 0.
\end{aligned}$$

It follows that, for every $(y,z) \in \left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right]^2$ such that $y \leq z$,

$$\phi_\omega(y,z) \leq \phi_\omega\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, z\right) \equiv \psi_{\omega,\bar{\omega}}(z).$$

Therefore,

$$M(\omega,\bar{\omega}) = \max_{z\in\left[\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega},\frac{1}{\bar{\omega}}\right]} \psi_{\omega,\bar{\omega}}(z).$$

Since

$$\psi''_{\omega,\bar{\omega}}(z) = \left(1 - \frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}\right)\frac{2\omega^2}{(1-\omega z)^3} > 0,$$

function $\psi_{\omega,\bar{\omega}}(.)$ is strictly convex. Therefore,

$$M(\omega,\bar{\omega}) = \max\left\{\phi_{\omega,\bar{\omega}}\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}\right), \phi_{\omega,\bar{\omega}}\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right)\right\}.$$

Since $\phi_{\omega,\bar{\omega}}(z,z) = 2(\omega-1)z < 0$ for every z, it follows that $M(\omega,\bar{\omega}) \leq 0$ if and only if $\zeta(\omega,\bar{\omega}) \leq 0$, where

$$\begin{aligned}
\zeta(\omega,\bar{\omega}) &\equiv \phi\left(\frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega}, \frac{1}{\bar{\omega}}\right), \\
&= \left(1 - \frac{\omega}{\bar{\omega}}\right)\frac{\frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}}{1 - \frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}} + \frac{\omega}{\bar{\omega}-\omega}\left(1 - \frac{\omega}{1-\omega}\frac{1-\bar{\omega}}{\bar{\omega}}\right) - \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega} - \frac{1}{\bar{\omega}}, \\
&= \frac{\omega(1-\bar{\omega})}{\bar{\omega}} + \frac{\omega}{(1-\omega)\bar{\omega}} - \frac{1-\bar{\omega}}{\bar{\omega}}\frac{1}{1-\omega} - \frac{1}{\bar{\omega}}, \\
&= \frac{1}{1-\omega} + \frac{\omega-2}{\bar{\omega}} - \omega.
\end{aligned}$$

For every $\omega \in (0,\bar{\omega})$,

$$\frac{\partial \zeta}{\partial \omega} = \frac{1}{(1-\omega)^2} + \frac{1}{\bar{\omega}} - 1 > 0.$$

Therefore, ζ is strictly increasing in ω on interval $(0,\bar{\omega})$. It follows that $M(\omega,\bar{\omega}) \leq 0$ for every $\omega \in (0, \bar{\omega})$ if and only if $\xi(\bar{\omega}) \leq 0$, where

$$\begin{aligned}\xi(\bar{\omega}) &\equiv \zeta(\bar{\omega}, \bar{\omega}), \\ &= \frac{1}{1-\bar{\omega}} + 1 - \bar{\omega} - \frac{2}{\bar{\omega}}.\end{aligned}$$

For every $\bar{\omega} \in (0,1)$,

$$\xi'(\bar{\omega}) = \frac{1}{(1-\bar{\omega})^2} + \frac{2}{(\bar{\omega})^2} - 1 > 0.$$

Therefore, ξ is strictly increasing on $(0,1)$. Since $\lim_{0^+} \xi = -\infty$ and $\lim_{1^-} \xi = +\infty$, there exists a unique threshold $\omega^* \in (0,1)$ such that $\xi(\bar{\omega}) \leq 0$ if and only if $\bar{\omega} \leq \omega^*$. Numerically, we find that $\omega^* \simeq 0.64$. □

We can now prove Lemma XIV:

Proof. Assume that $\bar{\omega}^f < \omega^*$ (or, equivalently, that $\bar{\mu}^f < \mu^* \simeq 2.78$). Splitting the sum in two terms, condition (xxv) can be rewritten as follows:

$$\begin{aligned}&\forall \omega^f \in (0, \bar{\omega}^f),\ \forall (x_k)_{k\in f} \in \left\{(x_k)_{k\in f} \in \mathbb{R}^f_{++} :\ \forall k \in f,\ \chi_k(x_k) > \omega^f\right\}, \\ &\frac{1}{2}\sum_{\substack{i,j\in f \\ i\neq j}} \gamma_i(x_i)\gamma_j(x_j)\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_j(x_j)}{1-\omega^f\theta_i(x_i)} + \omega^f\theta_j(x_j)\frac{1-\omega^f\rho_i(x_i)}{1-\omega^f\theta_j(x_j)} - \rho_i(x_i) - \rho_j(x_j)\right) \\ &+\left(\sum_{i\in f}\gamma_i(x_i)^2\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_i(x_i)}{1-\omega^f\theta_i(x_i)} - \rho_i(x_i)\right)\right) < 0.\end{aligned} \tag{xxvi}$$

Let us first show that the second sum in equation (xxvi) is strictly negative. Let $\omega^f \in (0, \bar{\omega}^f)$, $i \in f$ and x_i such that $\chi_i(x_i) > \omega^f$. Then,

$$\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_i(x_i)}{1-\omega^f\theta_i(x_i)} - \rho_i(x_i) \leq \omega^f\theta_i(x_i) - \rho_i(x_i) < 0,$$

where the first inequality follows from the fact that ρ_i is non-decreasing ($\theta_i(x_i) \leq \rho_i(x_i)$) and Lemma VIII ($1 - \omega^f\theta_i(x_i) > 0$).

Next, we turn our attention to the first sum in equation (xxvi). Let $\omega^f \in (0, \bar{\omega}^f)$ and $(x_k)_{k\in f}$ such that $\chi_k(x_k) > \omega^f$ for every $k \in f$. By Lemma XV,

$$\forall k \in f,\ \rho_k(x) \in \left[\frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}, \frac{1}{\bar{\omega}^f}\right].$$

In addition, as shown above, for every $k \in f$, $\theta_k(x_k) \leq \rho_k(x_k) \left(< \frac{1}{\omega^f}\right)$. Therefore,

$$\begin{aligned}
&\frac{1}{2} \sum_{\substack{i,j \in f \\ i \neq j}} \gamma_i(x_i)\gamma_j(x_j) \left(\omega^f \theta_i(x_i) \frac{1 - \omega^f \rho_j(x_j)}{1 - \omega^f \theta_i(x_i)} + \omega^f \theta_j(x_j) \frac{1 - \omega^f \rho_i(x_i)}{1 - \omega^f \theta_j(x_j)} - \rho_i(x_i) - \rho_j(x_j)\right) \\
&\leq \frac{1}{2} \sum_{\substack{i,j \in f \\ i \neq j}} \gamma_i(x_i)\gamma_j(x_j) \phi_{\omega^f, \bar{\omega}^f} \left(\rho_i(x_i), \rho_j(x_j)\right), \\
&\leq 0, \text{ by Lemma XVI.}
\end{aligned}$$

□

VII.4 Sufficiency of condition (c)

We prove a slightly more general result:

Lemma XVII. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that there exist $h^f \in \mathbb{R}_{++}^{\mathbb{R}_{++}}$, $c^f > 0$ and $(\alpha_k)_{k \in f} \in \mathbb{R}_{++}^f$ such that for every k in f, $c_k = c^f$, and for every $x > 0$, $h_k(x) = \alpha_k h^f(x)$. Assume in addition that ρ^f is non-decreasing. Then, condition* (xx) *holds.*

Proof. Let $k \in f$. It is straightforward to show that $\theta_k = \theta^f$, $\rho_k = \rho^f$, $\gamma_k = \alpha_k \gamma^f$, $\iota_k = \iota^f$, and $\chi_k = \chi^f$. In addition, $\nu_k = \nu^f$. Therefore, $r_k = r^f$. Condition (xx) is equivalent to

$$\forall \omega^f \in (0, \bar{\omega}^f), \; \left(\sum_{k \in f} \frac{\omega^f \theta^f}{1 - \omega^f \theta^f} \alpha_k \gamma\right) \left(\frac{1}{\sum_{k \in f} \alpha_k h^f} - \frac{\omega^f}{\sum_{k \in f} \alpha_k \gamma^f}\right) < 1,$$

where all functions are evaluated at $r^f\left(\frac{1}{1-\omega^f}, c^f\right)$. This is equivalent to

$$\frac{1 - \omega^f \rho^f}{1 - \omega^f \theta^f} \frac{\omega^f \theta^f}{\rho^f} < 1,$$

which clearly holds, since $\theta^f \leq \rho^f$. □

VII.5 Condition (b) when $\lim_\infty h_j \geq 0$

In this section, we extend condition (b) in Theorem 2 to cases where $\lim_\infty h_j$ is not necessarily equal to zero. We start with the following technical lemma:

Lemma XVIII. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ for every $j \in f$. Then, for every $k \in f$,*

$$S_k = \left\{\omega \in (0, \bar{\omega}^f) : \; \exists x > \underline{p}_k, \; \omega = \chi_k(x) = \frac{1}{\rho_k(x)}\right\}$$

contains at most one element. If S_k is empty, then, either $\chi_k(x)\rho_k(x) > 1$ for every $x > \underline{p}_k$, or $\chi_k(x)\rho_k(x) < 1$ for every $x > \underline{p}_k$. If, instead, $S_k = \{\hat{\omega}\}$, then, for every $x > \underline{p}_k$,

- $\theta_k(x) \leq \frac{1}{\hat{\omega}}$, *and*
- *if* $\rho_k(x) < \frac{1}{\hat{\omega}}$, *then* $\rho_k(x) \geq \frac{1-\hat{\omega}}{\hat{\omega}} \frac{1}{1-\chi_k(x)}$.

Proof. Let $k \in f$, and assume for a contradiction that S_k contains two distinct elements. There exist $x, y > \underline{p}_k$ such that $\chi_k(x)\rho_k(x) = 1$, $\chi_k(y)\rho_k(y) = 1$ and $\chi_k(x) \neq \chi_k(y)$. To fix ideas, assume $\chi_k(y) > \chi_k(x)$. Then, since χ_k is non-decreasing, $y > x$. Since ρ_k is non-decreasing, $\rho_k(x) \leq \rho_k(y)$. Therefore, $\chi_k(x)\rho_k(x) < \chi_k(y)\rho_k(y) = 1$, which is a contradiction.

Let $\kappa : x \in (\underline{p}_k, \infty) \mapsto \rho_k(x)\chi_k(x)$, and notice that κ is continuous and non-decreasing. If $S_k = \emptyset$, then, there is no x such that $\kappa(x) = 1$. Since κ is continuous, either $\kappa > 1$, or $\kappa < 1$.

Next, let $x > \underline{p}_k$. If $\rho_k(x) \leq \frac{1}{\hat{\omega}}$, then, $\theta_k(x) \leq \rho_k(x) \leq \frac{1}{\hat{\omega}}$. Assume instead that $\rho_k(x) > \frac{1}{\hat{\omega}}$. Let $\hat{x}$ such that $\chi_k(\hat{x}) = \hat{\omega} = \frac{1}{\rho_k(\hat{x})}$. Then, $\rho_k(x) > \rho_k(\hat{x}) = \frac{1}{\hat{\omega}}$ and, by monotonicity, $x > \hat{x}$. Therefore, $\chi_k(x) \geq \chi_k(\hat{x}) = \hat{\omega}$. Next, we claim that $\theta_k(x) \leq \frac{1}{\chi_k(x)}$. To see this, notice that $\iota_k(x) = x\frac{-h_k'(x)}{\gamma_k(x)}$. Therefore,

$$\begin{aligned}
\frac{\iota_k'(x)}{\iota_k(x)} &= \frac{1}{x} + \frac{h_k''(x)}{h_k'(x)} - \frac{\gamma_k'(x)}{\gamma_k(x)}, \\
&= \frac{1}{x}\left(1 - \iota_k(x) + \frac{\gamma_k'(x)}{h_k'(x)} x \frac{-h_k'(x)}{\gamma_k(x)}\right), \\
&= \frac{1}{x}\left(1 - \iota_k(x) + \frac{\iota_k(x)}{\theta_k(x)}\right).
\end{aligned}$$

Therefore,

$$\theta_k(x) = \frac{\iota_k(x)}{\iota_k(x) - 1 + x\frac{\iota_k'(x)}{\iota_k(x)}} \leq \frac{\iota_k(x)}{\iota_k(x) - 1} = \frac{1}{\chi_k(x)}.$$

Therefore, $\theta_k(x) \leq \frac{1}{\chi_k(x)} \leq \frac{1}{\hat{\omega}}$.

Next, assume that $\rho_k(x) < \frac{1}{\hat{\omega}}$. We know from the proof of Lemma XV that for every $t \in [x, \hat{x}]$,

$$\begin{aligned}
\frac{\rho_k'(t)}{\rho_k(t)} &= \frac{\iota_k'(t)}{\iota_k(t)} + \frac{\iota_k(t)}{t\rho_k(t)}\left(\rho_k(t)\chi_k(t) - 1\right), \\
&\leq \frac{\iota_k'(t)}{\iota_k(t)} + \frac{\iota_k(t)}{t\rho_k(t)}\left(\rho_k(\hat{x})\chi_k(\hat{x}) - 1\right), \text{ by monotonicity,} \\
&= \frac{\iota_k'(t)}{\iota_k(t)}, \text{ since } \rho_k(\hat{x})\chi_k(\hat{x}) = 1.
\end{aligned}$$

Integrating this inequality between x and $\hat{x}$, we obtain that $\frac{\rho_k(\hat{x})}{\rho_k(x)} \leq \frac{\iota_k(\hat{x})}{\iota_k(x)}$. Therefore,

$$\rho_k(x) \geq \rho_k(\hat{x})\frac{\iota_k(x)}{\iota_k(\hat{x})} = \frac{1-\hat{\omega}}{\hat{\omega}}\frac{1}{1-\chi_k(x)}. \qquad \square$$

Proposition III. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ for every $j \in f$ and that $\bar{\omega}^f \leq \omega^*$. Assume also, using the notation introduced in Lemma XVIII that, for every $i \in f$, $S_i = \{\hat{\omega}\}$. Then, condition* (xxv) *holds for firm f.*

Proof. As in the proof of Theorem XIV, the expression in condition (xxv) can be split in two terms (see equation (xxvi)). Since ρ_j is non-decreasing for every $j \in f$ and by Lemma VIII, the second sum is strictly negative. Next, we turn our attention to the first sum. Let $\omega^f \in (0, \bar{\omega}^f)$, $i, j \in f$, and x_i, x_j such that $\chi_i(x_i) > \omega^f$ and $\chi_j(x_j) > \omega^f$. We want to show that

$$\Psi = \omega^f \theta_i(x_i) \frac{1 - \omega^f \rho_j(x_j)}{1 - \omega^f \theta_i(x_i)} + \omega^f \theta_j(x_j) \frac{1 - \omega^f \rho_i(x_i)}{1 - \omega^f \theta_j(x_j)} - \rho_i(x_i) - \rho_j(x_j) \leq 0. \tag{xxvii}$$

To fix ideas, assume that $\rho_i(x_i) \leq \rho_j(x_j)$. If $\rho_i(x_i) \geq \frac{1}{\omega^f}$, then condition (xxvii) is clearly satisfied, since, by Lemma VIII, $1 - \omega^f \theta_i(x_i)$ and $1 - \omega^f \theta_j(x_i)$ are strictly positive. Assume instead that $\rho_i(x_i) < \frac{1}{\omega^f}$. Then, we claim that $\omega^f < \hat{\omega}$. Assume for a contradiction that $\hat{\omega} \leq \omega^f$. Since $S_i = \{\hat{\omega}\}$, there exists $\hat{x}_i > \underline{p}_i$ such that $\chi_i(\hat{x}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{x}_i)}$. Therefore, $\rho_i(x_i) < \rho_i(\hat{x}_i)$ and, by monotonicity, $x_i < \hat{x}_i$. Since χ_i is non-decreasing, it follows that

$$\omega^f < \chi_i(x_i) \leq \chi_i(\hat{x}_i) = \hat{\omega},$$

which is a contradiction. Therefore, $\omega^f < \hat{\omega}$.

We distinguish three cases. Assume first that $\rho_j(x_j) < \frac{1}{\hat{\omega}}$. Then, by Lemma XVIII,

$$\rho_k(x_k) \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \chi_k(x_k)} \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \omega^f},$$

for $k \in \{i, j\}$. In addition, $\frac{\theta_i(x_i)}{1 - \omega^f \theta_i(x_i)} \leq \frac{\rho_i(x_i)}{1 - \omega^f \rho_i(x_i)}$ and $\frac{\theta_j(x_j)}{1 - \omega^f \theta_j(x_j)} \leq \frac{\rho_j(x_j)}{1 - \omega^f \rho_j(x_j)}$. Therefore,

$$\Psi \leq \phi_{\omega^f, \hat{\omega}} \left(\rho_i(x_i), \rho_j(x_j)\right),$$

which, by Lemma XVI, is non-positive, since $\hat{\omega} < \bar{\omega}^f \leq \omega^*$.

Next, assume that $\rho_i(x_i) < \frac{1}{\hat{\omega}} \leq \rho_j(x_j)$. Then, by Lemma XVIII,

$$\rho_i(x_i) \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \chi_i(x_i)} \geq \frac{1 - \hat{\omega}}{\hat{\omega}} \frac{1}{1 - \omega^f},$$

and $\theta_j(x_j) \leq \frac{1}{\hat{\omega}}$. Therefore,

$$\Psi \leq \frac{\omega^f \theta_i(x_i)}{1 - \omega^f \theta_i(x_i)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1 - \frac{\omega^f}{\hat{\omega}}} (1 - \omega^f \rho_i(x_i)) - \rho_i(x_i) - \frac{1}{\hat{\omega}},$$

$$\leq \frac{\omega^f \rho_i(x_i)}{1-\omega^f \rho_i(x_i)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1-\frac{\omega^f}{\hat{\omega}}}(1-\omega^f \rho_i(x_i)) - \rho_i(x_i) - \frac{1}{\hat{\omega}},$$
$$= \phi_{\omega^f,\hat{\omega}}\left(\rho_i(x_i), \frac{1}{\hat{\omega}}\right),$$
$$\leq 0 \text{ by Lemma XVI.}$$

Finally, assume that $\rho_i(x_i) \geq \frac{1}{\hat{\omega}}$. By Lemma XVIII, $\theta_i(x_i) \leq \frac{1}{\hat{\omega}}$ and $\theta_j(x_j) \leq \frac{1}{\hat{\omega}}$. Therefore,

$$\Psi \leq \frac{\omega^f \theta_i(x_i)}{1-\omega^f \theta_i(x_i)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\omega^f \theta_j(x_j)}{1-\omega^f \theta_j(x_j)} \left(1 - \frac{\omega^f}{\hat{\omega}}\right) - \frac{1}{\hat{\omega}} - \frac{1}{\hat{\omega}},$$
$$\leq \frac{\frac{\omega^f}{\hat{\omega}}}{1-\frac{\omega^f}{\hat{\omega}}}\left(1 - \frac{\omega^f}{\hat{\omega}}\right) + \frac{\frac{\omega^f}{\hat{\omega}}}{1-\frac{\omega^f}{\hat{\omega}}}\left(1 - \frac{\omega^f}{\hat{\omega}}\right) - \frac{1}{\hat{\omega}} - \frac{1}{\hat{\omega}},$$
$$= \phi_{\omega^f,\hat{\omega}}\left(\frac{1}{\hat{\omega}}, \frac{1}{\hat{\omega}}\right),$$
$$\leq 0 \text{ by Lemma XVI.} \qquad \square$$

Condition $S_i = \{\hat{\omega}\}$ $\forall i$ in Proposition III may look a little bit arcane. The following corollary is easier to understand:

Corollary I. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ for every $j \in f$ and that $\bar{\omega}^f \leq \omega^*$. Assume also that there exist $h \in \mathbb{R}_{++}^{\mathbb{R}_{++}}$ and $(\alpha_k, \beta_k)_{k \in f} \in \left(\mathbb{R}_{++}^2\right)^f$ such that for every $k \in f$, for every $x > 0$, $h_k(x) = \alpha_k h(\beta_k x)$. Then, condition* (xxv) *holds for firm f.*

Proof. Let us first show that $S_i \subseteq S_j$ for all $i, j \in f$. Let $i, j \in f$. If S_i is empty, then, trivially, $S_i \subseteq S_j$. Assume instead that $S_i \neq \emptyset$, and let $\hat{\omega} \in S_i$. There exists $\hat{x}_i > \underline{p}_i$ such that

$$\chi_i(\hat{x}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{x}_i)}.$$

Since $h_i(x_i) = \alpha_i h(\beta_i x)$, it is straightforward to show that $\rho_i(\hat{x}_i) = \rho(\beta_i \hat{x}_i)$ and $\chi_i(\hat{x}_i) = \chi(\beta_i \hat{x}_i)$. Let $\hat{x}_j = \frac{\beta_i}{\beta_j}\hat{x}_i$. Then,

$$\chi_j\left(\hat{x}_j\right) = \chi\left(\beta_j \frac{\beta_i}{\beta_j}\hat{x}_i\right) = \chi_i(\hat{x}_i) = \hat{\omega} = \frac{1}{\rho_i(\hat{x}_i)} = \frac{1}{\rho(\beta_i \hat{x}_i)} = \frac{1}{\rho_j(\hat{x}_j)}.$$

Therefore, $\hat{\omega} \in S_j$, and $S_i \subseteq S_j$. It follows that $S_i = S_j$ for all $i, j \in f$.

If $S_i \neq \emptyset$, then, by Proposition III, condition (xxv) holds for firm f. Assume instead that $S_i = \emptyset$ for all i. Let $i \in f$. By Lemma XVIII, either $\chi_i(x_i)\rho_i(x_i) < 1$ for all x_i, or $\chi_i(x_i)\rho_i(x_i) > 1$ for all x_i. Assume first that $\chi_i(x_i)\rho_i(x_i) < 1$ for all x_i. Let $j \in f$ and

$x_j > \underline{p}_j$. Then,

$$\chi_j(x_j)\rho_j(x_j) = \chi_i\left(\frac{\beta_j}{\beta_i}x_j\right)\rho_i\left(\frac{\beta_j}{\beta_i}x_j\right) < 1.$$

Therefore, $\chi_j\rho_j < 1$ for every j in f. It follows that

$$\lim_{\infty}\rho_j \leq \lim_{\infty}\frac{1}{\chi_j} = \frac{1}{\bar{\omega}^f} < \infty.$$

Therefore, $\lim_\infty h_j = 0$ for every $j \in f$ (if $\lim_\infty h_j$ were strictly positive, then $\rho_j(x_j)$ would go to ∞ as x_j goes to ∞). By Lemma XIV, condition (xxv) holds for firm f.

Finally, assume that $\chi_i(x_i)\rho_i(x_i) > 1$ for all x_i. Then, using the same argument as above, $\chi_j\rho_j > 1$ for every $j \in f$. Let $i \in f$, and assume for a contradiction that $\underline{p}_i > 0$. Since $1/\chi_i$ is non-increasing, and since, by continuity, $\iota_i(\underline{p}_i) = 1$, it follows that $\lim_{\underline{p}_i^+}\frac{1}{\chi_i} = \infty$. Therefore, $\lim_{\underline{p}_i^+}\rho_i = \infty$, which is a contradiction, since ρ_i is non-decreasing. Therefore, $\underline{p}_i = 0$.

Assume for a contradiction that $\lim_{0^+}\iota_i = 1$. Then, using the same reasoning as in the previous paragraph, $\lim_{0^+}\rho_i = \infty$, which is again a contradiction, since ρ_i is non-decreasing. Therefore, $\lim_{0^+}\iota_i > 1$, and $\hat{\omega} \equiv \lim_{0^+}\chi_i$ is strictly positive. In addition, since

$$\chi_j(x) = \chi_i\left(\frac{\beta_j}{\beta_i}x\right),$$

$\lim_{0^+}\chi_j = \hat{\omega}$ for every $j \in f$. Notice that, for every $j \in f$, for every $x > 0$,

$$\rho_j(x) \geq \lim_{0^+}\rho_j \geq \lim_{0^+}\frac{1}{\chi_j} = \frac{1}{\hat{\omega}},$$

and that, by Lemma VIII,

$$\theta_j(x) \leq \frac{1}{\chi_j(x)} \leq \lim_{0^+}\frac{1}{\chi_j} = \frac{1}{\hat{\omega}}.$$

It follows that

$$\max_{i\in f}\sup\theta_i \leq \frac{1}{\hat{\omega}} \leq \min_{i\in f}\inf\rho_i,$$

i.e., condition (i) in Theorem 2 holds. By Lemma XIII, condition (xxv) is therefore satisfied for firm f. □

Proposition IV. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ for every $j \in f$, that $\bar{\omega}^f \leq \omega^*$, and that $\theta_k \leq \frac{1}{\bar{\omega}^f}$ for every k in f. Then, condition* (xxv) *holds for firm f.*

Proof. Let $i, j \in f$, $\omega^f \in (0, \bar{\omega}^f)$ and $x_i, x_j > 0$ such that $\chi_i(x_i) > \omega^f$ and $\chi_j(x_j) > \omega^f$. Define

$$\Psi = \frac{\omega^f\theta_i(x_i)}{1-\omega^f\theta_i(x_i)}\left(1-\omega^f\rho_j(x_j)\right) + \frac{\omega^f\theta_j(x_j)}{1-\omega^f\theta_j(x_j)}\left(1-\omega^f\rho_i(x_i)\right) - \rho_i(x_i) - \rho_j(x_j).$$

As in the previous proofs, all we need to do is show that $\Psi \le 0$. Assume first that $\rho_i(x_i) \ge \frac{1}{\bar{\omega}^f}$ and $\rho_j(x_j) \ge \frac{1}{\bar{\omega}^f}$. Then,

$$\max\left(\theta_i(x_i), \theta_j(x_j)\right) \le \min\left(\rho_i(x_i), \rho_j(x_j)\right).$$

Therefore, $\Psi < 0$.

Next, assume that $\rho_i(x_i) < \frac{1}{\bar{\omega}^f}$ and $\rho_j(x_j) \ge \frac{1}{\bar{\omega}^f}$. Then, we claim that

$$\rho_i(x_i) \ge \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}. \tag{xxviii}$$

To see this, assume first that $S_i = \{\hat{\omega}_i\}$, where $\hat{\omega}_i \in (0, \bar{\omega}^f)$. Since $\rho_i(x_i) < \frac{1}{\bar{\omega}^f} < \frac{1}{\hat{\omega}^i}$, by Lemma XVIII,

$$\rho_i(x_i) \ge \frac{1-\hat{\omega}_i}{\hat{\omega}_i}\frac{1}{1-\chi_i(x_i)} \ge \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}.$$

Assume instead that $S_i = \emptyset$. By Lemma XVIII, either $\chi_i\rho_i < 1$ or $\chi_i\rho_i > 1$. If $\chi_i\rho_i > 1$, then we know from the proof of Corollary I that

$$\rho_i \ge \sup\frac{1}{\chi_i} \ge \frac{1}{\bar{\omega}^f}.$$

This contradicts our assumption that $\rho_i(x_i) < \frac{1}{\bar{\omega}^f}$. If, instead, $\chi_i\rho_i < 1$, then we know from the proof of Corollary I that $\lim_\infty h_i = 0$. Therefore, by Lemma XV, inequality (xxviii) holds.

Therefore,

$$\begin{aligned}
\Psi &\le \frac{\omega^f\theta_i(x_i)}{1-\omega^f\theta_i(x_i)}\left(1-\frac{\omega^f}{\bar{\omega}^f}\right) + \frac{\frac{\omega^f}{\bar{\omega}^f}}{1-\frac{\omega^f}{\bar{\omega}^f}}\left(1-\omega^f\rho_i(x_i)\right) - \rho_i(x_i) - \frac{1}{\bar{\omega}^f},\\
&\le \frac{\omega^f\rho_i(x_i)}{1-\omega^f\rho_i(x_i)}\left(1-\frac{\omega^f}{\bar{\omega}^f}\right) + \frac{\frac{\omega^f}{\bar{\omega}^f}}{1-\frac{\omega^f}{\bar{\omega}^f}}\left(1-\omega^f\rho_i(x_i)\right) - \rho_i(x_i) - \frac{1}{\bar{\omega}^f},\\
&= \phi_{\omega^f,\bar{\omega}^f}\left(\rho_i(x_i), \frac{1}{\bar{\omega}^f}\right),\\
&\le 0 \text{ by Lemma XVI.}
\end{aligned}$$

Finally, assume that $\rho_i(x_i) < \frac{1}{\bar{\omega}^f}$ and $\rho_j(x_j) < \frac{1}{\bar{\omega}^f}$. Then, as above,

$$\rho_k(x_k) \ge \frac{1-\bar{\omega}^f}{\bar{\omega}^f}\frac{1}{1-\omega^f}$$

for $k \in \{i, j\}$. Therefore,

$$\Psi \le \phi_{\omega^f,\bar{\omega}^f}\left(\rho_i(x_i), \rho_j(x_j)\right),$$

which is non-positive by Lemma XVI. □

Corollary II. *Assume that $\bar{\mu}^f = \bar{\mu}_j$ for every $f \in \mathcal{F}$ and $j \in f$. Let $f \in \mathcal{F}$. Assume that ρ_j is non-decreasing on $\left(\underline{p}_j, \infty\right)$ for every $j \in f$, that $\bar{\omega}^f \leq \omega^*$, and that θ_k is non-decreasing for every k in f. Then, condition* (xxv) *holds for firm f.*

Proof. Let $k \in f$. Since θ_k is non-increasing, for every $x > \underline{p}_k$,

$$\theta_k(x) \leq \sup \theta_k = \lim_{\infty} \theta_k \leq \lim_{\infty} \frac{1}{\chi_k} = \frac{1}{\bar{\omega}^f},$$

where the second inequality follows from Lemma VIII. Therefore, by Proposition IV, condition (xxv) holds for firm f. □

VII.6 Proof of Proposition 8.

Proof. Let $j \in f$. Then, for all $x > 0$,

$$\begin{aligned}
h_j'(x) &= \alpha_j \beta_j h'(\beta_j x + \delta_j) < 0, \\
h_j''(x) &= \alpha_j \beta_j^2 h''(\beta_j x + \delta_j) > 0, \\
\gamma_j(x) &= \alpha_j \gamma(\beta_j x + \delta_j), \\
\gamma_j'(x) &= \alpha_j \beta_j \gamma'(\beta_j x + \delta_j), \\
\rho_j(x) &= \rho(\beta_j x + \delta_j) + \frac{\epsilon_j}{\alpha_j \gamma(\beta_j x + \delta_j)} \geq \rho(\beta_j x + \delta_j), \\
\theta_j(x) &= \theta(\beta_j x + \delta_j), \\
\iota_j(x) &= \frac{\beta_j x}{\beta_j x + \delta_j} \iota(\beta_j x + \delta_j).
\end{aligned}$$

Therefore, h_j is positive, decreasing and log-convex, ι_j is non-decreasing whenever ι_j is > 1, and $\bar{\mu}_j = \lim_\infty \iota$. In addition, for every $x > \underline{p}_j$,

$$1 < \iota_j(x) \leq \iota(\beta_j x + \delta_j).$$

Therefore, $\beta_j x + \delta_j > \underline{p}$, and

$$\theta_j(x) \leq \sup_{y > \underline{p}} \theta(y).$$

It follows that $\sup_{y > \underline{p}_j} \theta_j(y) \leq \sup_{y > \underline{p}} \theta(y)$. Using the same reasoning, we also obtain that $\inf_{y > \underline{p}_j} \rho_j(y) \geq \inf_{y > \underline{p}} \rho(y)$. Therefore,

$$\begin{aligned}
\max_{j \in f} \sup_{x > \underline{p}_j} \theta_j(x) &\leq \max_{j \in f} \sup_{x > \underline{p}} \theta(x), \\
&\leq \sup_{x > \underline{p}} \theta(x), \\
&\leq \inf_{x > \underline{p}} \rho(x),
\end{aligned}$$

$$\le \min_{j\in f} \inf_{x>\underline{p}} \rho(x),$$

$$\le \min_{j\in f} \inf_{x>\underline{p}_j} \rho_j(x). \qquad \square$$

VII.7 Proof of Proposition 9

In this section, we let $m^f(H,(c_j)_{j\in f})$ be firm f's fitting-in function when its costs are given by $(c_j)_{j\in f}$. It is straightforward to adapt the proof of Lemma H to show that m^f is non-increasing in $(c_j)_{j\in f}$, and that

$$\lim_{c^f\to\infty} m^f\left(H,\left(c^f,\dots,c^f\right)\right) = 1.$$

We introduce the following notation: For every $f \in \mathcal{F}$, put $\underline{\mu}^f = \min_{j\in f} \bar{\mu}_j$ and $\underline{\omega}^f = \frac{\underline{\mu}^f - 1}{\underline{\mu}^f}$ (or $\underline{\omega}^f = 1$ if $\underline{\mu}^f = \infty$). For every $\underline{c} > 0$, define

$$\underline{H}(\underline{c}) = \min_{f\in\mathcal{F}} \inf\left\{H > 0 : \quad m^f(H,(\underline{c},\dots,\underline{c})) < \underline{\mu}^f\right\}.$$

By Lemma H, $\underline{H}(\underline{c})$ is finite, and $m^f(H,(\underline{c},\dots,\underline{c})) < \underline{\mu}^f$ for all $f\in\mathcal{F}$ whenever $H > \underline{H}(\underline{c})$. In addition, since m^f is decreasing in $(c_j)_{j\in f}$, $m^f\left(H,(c_j)_{j\in f}\right) < \underline{\mu}^f$ for all $H > \underline{H}(\underline{c})$, $f\in\mathcal{F}$ and $(c_j)_{j\in f} \in [\underline{c},\infty)^f$. Note also that $\underline{H}$ is non-increasing in $\underline{c}$, and that $\lim_{\underline{c}\to\infty} \underline{H}(\underline{c}) = 0$.

We prove the following preliminary technical lemma:

Lemma XIX. *Let $\underline{c} > 0$. If, for every $f\in\mathcal{F}$,*

$$\begin{gathered}\forall \omega^f \in (0,\underline{\omega}^f),\ \forall (x_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\frac{1}{1-\omega^f},\underline{c}\right),\infty\right), \\ \left(\sum_{k\in f}\frac{\omega^f\theta_k(x_k)}{1-\omega^f\theta_k(x_k)}\gamma_k(x_k)\right)\left(\frac{1}{\sum_{k\in f} h_k(x_k)} - \frac{\omega^f}{\sum_{k\in f}\gamma_k(x_k)}\right) < 1,\end{gathered} \tag{xxix}$$

or, equivalently, if

$$\begin{gathered}\forall \omega^f \in (0,\underline{\omega}^f),\ \forall (x_k)_{k\in f} \in \prod_{k\in f}\left[r_k\left(\frac{1}{1-\omega^f},\underline{c}\right),\infty\right), \\ \sum_{i,j\in f}\gamma_i(x_i)\gamma_j(x_j)\left(\omega^f\theta_i(x_i)\frac{1-\omega^f\rho_j(x_j)}{1-\omega^f\theta_i(x_i)} - \rho_i(x_i)\right) < 0,\end{gathered} \tag{xxx}$$

then, for every $(c_j)_{j\in\mathcal{N}} \in [\underline{c},\infty)^{\mathcal{N}}$, pricing game $((h_j)_{j\in\mathcal{N}},\mathcal{F},(c_j)_{j\in\mathcal{N}})$ has at most one equilibrium aggregator level in $(\underline{H}(\underline{c}),\infty)$.

Proof. The proof is exactly the same as the proof of Lemma XI. $\square$

We can now prove Proposition 9:

Proof. We only prove the first bullet point. The proof of the second bullet point is similar, and therefore omitted.

Let $\underline{H}^0 > 0$, and $H^0 \geq \underline{H}^0$. Recall that pricing game $((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ with outside option H^0 is equivalent to pricing game $((h_j^{H^0})_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ with outside option 0, where

$$h_j^{H^0} = h_j + \frac{H^0}{|\mathcal{N}|} \quad \forall j \in \mathcal{N}.$$

Note that, for every $H^0 \geq \underline{H}^0$ and $j \in \mathcal{N}$, $\rho_j^{H^0} \geq \rho_j^{\underline{H}^0}$ and $\lim_{p\to\infty} \rho_j^{H^0} = \infty$.

Fix some $c > \max_{j\in\mathcal{N}} \underline{p}_j$. For every $j \in \mathcal{N}$ and $x \geq c$,[7]

$$\theta_j(x) \leq \frac{1}{\chi_j(x)} \leq \frac{1}{\chi_j(c)} \leq \max_{k\in\mathcal{N}} \frac{1}{\chi_k(c)} \equiv \bar{\theta},$$

where the first inequality follows by Lemma VIII. Since $\lim_\infty \rho_j^{\underline{H}^0} = \infty$ for every $j \in \mathcal{N}$, there exists $c' > c$ such that, for every $j \in \mathcal{N}$, $\rho_j^{\underline{H}^0}(x) \geq \bar{\theta}$ whenever $x \geq c'$. Therefore, for every $H^0 \geq \underline{H}^0$, $f \in \mathcal{F}$, $i, j \in f$, $x_i \geq c'$ and $x_j \geq c'$, $\rho_i^{H^0}(x_i) \geq \theta_j^{H^0}(x_j)$, and, in particular,

$$\forall \omega^f \in (0, \underline{\omega}^f), \quad \frac{\omega^f \theta_i^{H^0}(x_i)}{1 - \omega^f \theta_i^{H^0}(x_i)} \left(1 - \omega^f \rho_j^{H^0}(x_j)\right) - \rho_i^{H^0}(x_i) < 0.$$

Therefore, condition (xxx) holds for lower bound c' (or higher), and, for every $H^0 \geq \underline{H}^0$ and $(c_j)_{j\in\mathcal{N}} \in [c', \infty)^{\mathcal{N}}$, pricing game $((h_j^{H^0})_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ has at most one equilibrium aggregator level in $(\underline{H}(c'), \infty)$.

Next, choose $c'' > 0$ such that $\underline{H}(c'') < \underline{H}^0$. Since $\lim_\infty \underline{H} = 0$, such a c'' exists. Put $\underline{c} = \max(c', c'')$. Since $\underline{H}(\cdot)$ is non-increasing, $\underline{H}(\underline{c}) < \underline{H}^0$. Combining this with our previous findings, we can conclude that for every $H^0 \geq \underline{H}^0$ and $(c_j)_{j\in\mathcal{N}} \in [\underline{c}, \infty)^{\mathcal{N}}$, pricing game $((h_j)_{j\in\mathcal{N}}, \mathcal{F}, (c_j)_{j\in\mathcal{N}})$ with outside option H^0 has at most one equilibrium aggregator level in (H^0, ∞). Since this pricing game has an equilibrium (Theorem 1), and since no equilibrium aggregator level can be less than H^0, it follows that this pricing game has a unique equilibrium. □

VII.8 Establishing Equilibrium Uniqueness Using an Index Approach

The reader may wonder whether we could obtain weaker uniqueness conditions by using more standard approaches. Uniqueness of a fixed point is usually established by using the contraction mapping approach, the univalence approach or the index (Poincaré-Hopf) approach. It is well known that the index approach is more general than the others, and that it provides an "almost if and only if" condition for uniqueness. We will therefore focus on

[7] Since neither θ_j nor χ_j depend on H^0, we drop superscript H^0 to ease notation.

the index approach. Since we will be working with matrices, we will sometimes assume that $\mathcal{F} = \{1, \dots, F\}$, and that firm f's set of products is $\mathcal{N}^f$.

We know that establishing uniqueness in the pricing game is equivalent to establishing uniqueness in the auxiliary game in which firms are simultaneously choosing their μ^f's. We also know that a profile $\mu = (\mu^f)_{f\in\mathcal{F}}$ is an equilibrium of the auxiliary game if and only if for every $f \in \mathcal{F}$,

$$\phi^f(\mu) \equiv (\mu^f - 1)\left(\left(\sum_{k\in\mathcal{N}^f} h_k\right) + \left(\sum_{\substack{g\in\mathcal{F}\\ g\neq f}} \sum_{k\in\mathcal{N}^f} h_k\right)\right) - \mu^f \sum_{k\in\mathcal{N}^f} \gamma_k = 0.$$

Therefore, all we need to do is show that map ϕ has a unique zero. By the index theorem, this holds if the determinant of the Jacobian matrix of ϕ evaluated at μ is strictly positive whenever $\phi(\mu) = 0$. We have shown in the proof of Lemma F that

$$\frac{\partial \phi^f}{\partial \mu^f} = \sum_{f\in\mathcal{F}} \sum_{k\in\mathcal{N}^f} h_k \equiv H(\mu).$$

Moreover, if $g \neq f$, then

$$\frac{\partial \phi^f}{\partial \mu^g} = (\mu^f - 1) \sum_{k\in\mathcal{N}^g} r'_k h'_k.$$

Therefore,

$$\begin{aligned}
\det J(\phi) &= \begin{vmatrix} H(\mu) & (\mu_1 - 1)\sum_{k\in\mathcal{N}^2} r'_k h'_k & \cdots & (\mu_1 - 1)\sum_{k\in\mathcal{N}^F} r'_k h'_k \\ (\mu_2 - 1)\sum_{k\in\mathcal{N}^1} r'_k h'_k & H(\mu) & \cdots & (\mu_2 - 1)\sum_{k\in\mathcal{N}^F} r'_k h'_k \\ \vdots & \vdots & \ddots & \vdots \\ (\mu^F - 1)\sum_{k\in\mathcal{N}^1} r'_k h'_k & (\mu^F - 1)\sum_{k\in\mathcal{N}^2} r'_k h'_k & \cdots & H(\mu) \end{vmatrix}, \\
&= \left(\prod_{f\in\mathcal{F}} (\mu^f - 1) \sum_{k\in\mathcal{N}^f} r'_k h'_k\right) \det \mathcal{M}\left(\left(1 + \frac{H(\mu)}{(\mu^f - 1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}\right)_{1\leq f\leq F}\right),
\end{aligned}$$

where the second line has been obtained by dividing row f by $\mu^f - 1$ and dividing column f by $\sum_{k\in\mathcal{N}^f} r'_k h'_k$ for every f in $\{1, \dots, F\}$ and by using the F-linearity of the determinant. By Lemma I,

$$\begin{aligned}
\det(J(\phi)) = &\left(\prod_{f\in\mathcal{F}} (\mu^f - 1) \sum_{k\in\mathcal{N}^f} r'_k h'_k\right) (-1)^F \left(\left(\prod_{f\in\mathcal{F}} \left(1 + \frac{H(\mu)}{(\mu^f - 1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}\right)\right)\right. \\
&\left. - \sum_{g\in\mathcal{F}} \prod_{f\neq g} \left(1 + \frac{H(\mu)}{(\mu^f - 1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}\right)\right),
\end{aligned}$$

$$= \left(\prod_{f\in\mathcal{F}}(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k h'_k\right)(-1)^F\left(\prod_{f\in\mathcal{F}}\left(1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}\right)\right)$$
$$\times\left(1-\sum_{f\in\mathcal{F}}\frac{1}{1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}}\right),$$
$$= \underbrace{\left(\prod_{f\in\mathcal{F}}\left(H(\mu)+(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)\right)\right)}_{>0}\left(1-\sum_{f\in\mathcal{F}}\frac{1}{1+\frac{H(\mu)}{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}}\right).$$

Therefore, we need to show that

$$\sum_{f\in\mathcal{F}}\frac{\frac{\mu^f-1}{H(\mu)}\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{1+\frac{\mu^f-1}{H(\mu)}\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}<1 \tag{xxxi}$$

whenever $\phi(\mu)=0$. Notice that

$$\begin{aligned}
\text{(xxxi)} &\iff \sum_{f\in\mathcal{F}}\left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{1+\frac{\mu^f-1}{H(\mu)}\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}-\sum_{k\in\mathcal{N}^f} h_k\right)<0\\
&\iff \sum_{f\in\mathcal{F}}\left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{1+\frac{(\mu^f-1)^2}{\mu^f}\frac{\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{\sum_{k\in\mathcal{N}^f}\gamma_k}}-\sum_{k\in\mathcal{N}^f} h_k\right)<0,\ \text{since } \phi(\mu)=0,\\
&\iff \sum_{f\in\mathcal{F}}\left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{1+\frac{\mu^f-1}{\mu^f}\frac{\sum_{k\in\mathcal{N}^f} r'_k\left((\mu^f-1)(-h'_k)-\mu^f(-\gamma'_k)+\mu^f(-\gamma'_k)\right)}{\sum_{k\in\mathcal{N}^f}\gamma_k}}-\sum_{k\in\mathcal{N}^f} h_k\right)<0,\\
&\iff \sum_{f\in\mathcal{F}}\left(\frac{(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{1-\frac{\mu^f-1}{\mu^f}+(\mu^f-1)\frac{\sum_{k\in\mathcal{N}^f} r'_k(-\gamma'_k)}{\sum_{k\in\mathcal{N}^f}\gamma_k}}-\sum_{k\in\mathcal{N}^f} h_k\right)<0,\ \text{by Lemma D},\\
&\iff \sum_{f\in\mathcal{F}}\left(\frac{\mu^f(\mu^f-1)\sum_{k\in\mathcal{N}^f} r'_k(-h'_k)}{1+\mu^f(\mu^f-1)\frac{\sum_{k\in\mathcal{N}^f} r'_k(-\gamma'_k)}{\sum_{k\in\mathcal{N}^f}\gamma_k}}-\sum_{k\in\mathcal{N}^f} h_k\right)<0,\\
&\iff \Omega'\left(H(\mu)\right)<0 \text{ (see the proof of Lemma X).}
\end{aligned}$$

Therefore, the index approach gives us the exact same condition as the aggregative games approach.

VIII CES and MNL Demands: Type Aggregation and Algorithm

VIII.1 Formulas for m' and S' and Preliminary Lemmas

Applying the implicit function theorem to equations (13) and (14) yields:

$$\text{(CES)} \quad m'(x) = \frac{\frac{\sigma-1}{\sigma} m(x)^2 \left(1 - \frac{m(x)}{\sigma}\right)^{\sigma-1}}{1 + \left(\frac{\sigma-1}{\sigma}\right)^2 m(x)^2 x \left(1 - \frac{m(x)}{\sigma}\right)^{\sigma-2}}, \tag{xxxii}$$

$$\text{(MNL)} \quad m'(x) = \frac{m(x)^2 \, \mathrm{e}^{-m(x)}}{1 + m(x)^2 x \, \mathrm{e}^{-m(x)}}. \tag{xxxiii}$$

Let $\alpha = (\sigma - 1)/\sigma$ in the CES case and $\alpha = 1$ in the MNL case. Note that $m = \sigma/(\sigma - (\sigma - 1)S)$ in the CES case, and $m = 1/(1 - S)$ in the MNL case. Therefore, in both cases, $m = 1/(1 - \alpha S)$, $S = \frac{1}{\alpha}\frac{m-1}{m}$, and $S' = \frac{m'}{\alpha m^2}$. This implies in particular that

$$\text{(CES)} \quad \frac{1}{\alpha}\frac{m(x) - 1}{m(x)} = S(x) = x\left(1 - \frac{m(x)}{\sigma}\right)^{\sigma-1},$$

$$\text{(MNL)} \quad \frac{m(x) - 1}{m(x)} = S(x) = x \, \mathrm{e}^{-m(x)}.$$

This allows us to obtain expressions for $S'(x)$, which do not explicitly depend on terms $(1 - m(x)/\sigma)^{\sigma-1}$, $(1 - m(x)/\sigma)^{\sigma-2}$ and $\mathrm{e}^{-m(x)}$:

$$\text{(CES)} \quad xS'(x) = \frac{m(x) - 1}{\frac{\sigma-1}{\sigma} m(x)\left(1 + \frac{\sigma-1}{\sigma}\frac{m(x)}{1-m(x)/\sigma}(m(x) - 1)\right)}, \tag{xxxiv}$$

$$\text{(MNL)} \quad xS'(x) = \frac{m(x) - 1}{m(x)\left(1 + m(x)(m(x) - 1)\right)}. \tag{xxxv}$$

Formulas (xxxiv) and (xxxv) are used at the end of Section 5.2.

Next, we use the fact that $m = 1/(1 - \alpha S)$ to replace $m(x)$ in the right-hand side of equations (xxxiv) and (xxxv). In the MNL case, we have that:

$$xS'(x) = \frac{S(x)}{1 + m^2(x)S(x)} = \frac{S(x)}{1 + \frac{S(x)}{(1-S(x))^2}} = \frac{S(x)(1 - S(x))^2}{1 - S(x) + S(x)^2}.$$

In the CES case, we have that:

$$xS'(x) = \frac{S(x)}{1 + \alpha^2 m^2(x)\frac{S(x)}{1-m(x)/\sigma}},$$

$$= \frac{S(x)}{1 + \alpha^2 \frac{1}{(1-\alpha S(x))^2} \frac{S(x)(1-\alpha S(x))}{1-S(x)}},$$
$$= \frac{S(x)}{1 + \alpha \frac{S(x)}{(1-S(x))(1-\alpha S(x))}},$$
$$= \frac{S(x)(1-S(x))(1-\alpha S(x))}{1 - S(x) + \alpha S^2(x)}.$$

Therefore, in both cases:

$$xS' = \frac{S(x)}{1 + \alpha \frac{S(x)}{(1-S(x))(1-\alpha S(x))}}, \tag{xxxvi}$$
$$= \frac{S(x)(1-S(x))(1-\alpha S(x))}{1 - S(x) + \alpha S^2(x)}. \tag{xxxvii}$$

Let $\varepsilon(x) = xS'(x)/S(x)$ be the elasticity of S. We prove the following technical lemmas:

Lemma XX. *$\varepsilon' < 0$.*

Proof. Using equation (xxxvi), we see that

$$\varepsilon(x) = \frac{1}{1 + \alpha \frac{S(x)}{(1-S(x))(1-\alpha S(x))}}.$$

Since $S' > 0$, it follows that $\varepsilon' < 0$. □

Lemma XXI. *$S'' < 0$. Therefore, S is strictly subadditive.*

Proof. Using equation (xxxvi) and the fact that $S(x) = x(1 - m(x)/\sigma)^{\sigma-1}$ in the CES case and $m(x) = x\exp(-m(x))$ in the MNL case, we see that

$$\text{(CES)} \quad S'(x) = \frac{\left(1 - \frac{m(x)}{\sigma}\right)^{\sigma-1}}{1 + \alpha \frac{S(x)}{(1-S(x))(1-\alpha S(x))}},$$
$$\text{(MNL)} \quad S'(x) = \frac{\mathrm{e}^{-m(x)}}{1 + m(x)^2 S(x)}.$$

Since $m' > 0$ and $S' > 0$, it follows that $S'' < 0$.

Let $y > 0$, and define $\xi : x \in \mathbb{R}_{++} \mapsto S(x+y) - S(x) - S(y)$. Note that $\lim_0 \xi = 0$, and that

$$\xi'(x) = S'(x+y) - S'(x) < 0,$$

since $S'' < 0$. Therefore, ξ is strictly decreasing, and $\xi < 0$. □

VIII.2 Proof of Proposition 11

Proof. The fact that $m' > 0$, $S' > 0$, and $\pi'(= m') > 0$ can be seen by inspecting equation (xxxii), (xxxiii), and (xxxvi). This proves point (i).

Applying the implicit function theorem to equation $\Omega(H) = 1$ yields:

$$\frac{dH^*}{dT^f} = \frac{S'\left(\frac{T^f}{H^*}\right)}{\sum_{g\in\mathcal{F}} \frac{T^g}{H^*} S'\left(\frac{T^g}{H^*}\right)} > 0. \tag{xxxviii}$$

Next, notice that

$$\frac{d\left(\frac{T^f}{H^*}\right)}{dT^f} = \frac{1}{H^*}\left(1 - \frac{T^f}{H^*}\frac{dH^*}{dT^f}\right) = \frac{1}{H^*}\left(1 - \frac{\frac{T^f}{H^*} S'\left(\frac{T^f}{H^*}\right)}{\sum_{g\in\mathcal{F}} \frac{T^g}{H^*} S'\left(\frac{T^g}{H^*}\right)}\right) > 0,$$

and that, for $g \neq f$,

$$\frac{d\left(\frac{T^g}{H^*}\right)}{dT^g} = -\frac{T^g}{H^{*2}}\frac{dH^*}{dT^f} < 0.$$

Therefore, points (ii) and (iii) follow immediately by applying the chain rule.

Next, we turn our attention to point (iv). Let $x^g = T^g/H^*$ for every g. Social welfare is given by

$$W^* = \log H^* + \sum_{g\in\mathcal{F}} (m(x^g) - 1).$$

Therefore,

$$\begin{aligned}
\frac{dW^*}{dT^f} &= \frac{1}{H^*}\left(\frac{dH^*}{dT^f}\left(1 - \sum_{g\in\mathcal{F}} x^g m'(x^g)\right) + m'(x^f)\right), \\
&= \frac{1}{H^*}\left(\frac{S'(x^f)}{\sum_{g\in\mathcal{F}} x^g S'(x^g)}\left(1 - \sum_{g\in\mathcal{F}} x^g \alpha \frac{S'(x^g)}{(1-\alpha S(x^g))^2}\right) + \alpha\frac{S'(x^f)}{(1-\alpha S(x^f))^2}\right), \\
&= \frac{S'(x^f)}{H^* \sum_{g\in\mathcal{F}} x^g S'(x^g)}\left(1 + \alpha \sum_{g\in\mathcal{F}} x^g S'(x^g)\left(\frac{1}{(1-\alpha S(x^f))^2} - \frac{1}{(1-\alpha S(x^g))^2}\right)\right), \\
&= \frac{S'(x^f)}{H^* \sum_{g\in\mathcal{F}} x^g S'(x^g)}\left(1 + \alpha \sum_{g\in\mathcal{F}} \frac{s^g(1-s^g)(1-\alpha s^g)}{1 - s^g + \alpha (s^g)^2}\left(\frac{1}{(1-\alpha s^f)^2} - \frac{1}{(1-\alpha s^g)^2}\right)\right), \\
&> \frac{S'(x^f)}{H^* \sum_{g\in\mathcal{F}} x^g S'(x^g)}\left(1 + \sum_{g\in\mathcal{F}} \underbrace{\alpha \frac{s^g(1-s^g)(1-\alpha s^g)}{1 - s^g + \alpha (s^g)^2}\left(1 - \frac{1}{(1-\alpha s^g)^2}\right)}_{\equiv \psi_\alpha(s^g)}\right),
\end{aligned}$$

where the second line follows from equation (xxxviii) and the fact that $m = \frac{1}{1-\alpha S}$, and the fourth line follows from equation (xxxvii).

If we can show that $1+\sum_{i=1}^n \psi_\alpha(s_i) \geq 0$ for every $\alpha \in (0,1]$, $n \geq 2$, and $(s_i)_{1\leq i\leq n} \in [0,1)^n$ such that $\sum_{i=1}^n s_i = 1$, then we are done. Routine calculations show that $\psi_\alpha(s) \geq \psi_1(s) \equiv \psi(s)$ for every s. Therefore, all we need to do is show that $1 + \sum_{i=1}^n \psi(s_i) \geq 0$ for every $n \geq 2$ and $(s_i)_{1\leq i\leq n} \in [0,1)^n$ such that $\sum_{i=1}^n s_i = 1$. Note that $\psi(s) = s^2(s-2)/(1-s+s^2)$. Routine calculations show that:

(i) ψ is concave on $[0, 1/2]$.

(ii) $\psi(0) = 0$.

(iii) $\psi(s) + \psi(1-s) = -1$ for every $s \in [0,1]$.

(iv) $\psi(s) > -s$ (resp. $\psi(s) < -s$) if and only if $s < 1/2$ (resp. $s > 1/2$).

By point (iv), if $s_i \leq 1/2$ for every i, then $1+\sum_{i=1}^n \psi(s_i) \geq 0$. Next, let $(s_i)_{1\leq i\leq n}$ such that $s_i > 1/2$ for some i. Assume without loss of generality that $s_n > 1/2$. Then, $\sum_{i=1}^{n-1} s_i < 1/2$. We claim that

$$\sum_{i=1}^{n-1} \psi(s_i) \geq \psi\left(\sum_{i=1}^{n-1} s_i\right). \qquad \text{(xxxix)}$$

To see this, let $x, y \in [0, 1/2]$ such that $x + y \leq 1/2$, and define

$$\xi : t \in [0, y] \mapsto \psi(x+t) - \psi(x) - \psi(t).$$

By point (ii), $\xi(0) = 0$. By point (i), $\xi' \leq 0$. Therefore, $\xi(t) \leq 0$ for every t. In particular, $\psi(x+y) \leq \psi(x) + \psi(y)$. Property (xxxix) follows by induction on n. Therefore,

$$1 + \sum_{i=1}^n \psi(s_i) \geq 1 + \psi\left(\sum_{i=1}^{n-1} s_i\right) + \psi(s_n) = 1 + \psi(1-s_n) + \psi(1-s_n) = 0,$$

where the last equality follows by point (iii). □

IX Comparative Statics

IX.1 Proof of Proposition 4

Proof. The first part of the proposition follows immediately from equation (2), Theorem 1 and Lemma H.

Next, we prove that largest and smallest (in terms of the value of H) equilibria exist. If there is a finite number of equilibrium aggregators, then this is trivial. Next, assume that there is an infinite number of equilibria. We have shown in the proof of Lemma I that $\Omega(H) > 1$ for H low enough and $\Omega(H) < 1$ for H high enough. Therefore, the set of equilibrium aggregators is contained in a closed interval $[\underline{H}, \overline{H}]$, with $\underline{H} > 0$. Put

$$\overline{H}^* \equiv \sup \left\{ H \in [\underline{H}, \overline{H}] : \quad \Omega(H) = 1 \right\}.$$

Let $(H^n)_{n \geq 0}$ be a sequence such that $\Omega(H^n) = 1$ for all n and $H^n \underset{n \to \infty}{\longrightarrow} \overline{H}^*$. Since Ω is continuous on $[\underline{H}, \overline{H}]$, we can take limits and obtain that $\Omega(\overline{H}^*) = 1$. Therefore,

$$\overline{H}^* = \max \left\{ H \in [\underline{H}, \overline{H}] : \quad \Omega(H) = 1 \right\}$$

is the highest equilibrium aggregator level. The existence of a lowest equilibrium aggregator follows from the same line of argument. □

IX.2 Proof of Proposition 5

Proof. Let $H^0 > 0$. Given outside option $H^0 \geq 0$, $H > 0$ is an equilibrium aggregator level if and only if $\Omega(H, H^0) = 1$, where

$$\Omega(H, H^0) = \frac{H^0 + \sum_{f \in \mathcal{F}} \sum_{j \in f} h_j \left(r_j \left(m^f(H) \right) \right)}{H}.$$

Let $H^{0\prime} > H^0 \geq 0$, and note that $\Omega(H, H^{0\prime}) > \Omega(H, H^0)$ for all $H > 0$. Let $\overline{H}$ and $\underline{H}$ (resp. $\overline{H}'$ and $\underline{H}'$) be the highest and lowest equilibrium aggregator levels when the outside option is H^0 (resp. $H^{0\prime}$). We know from the proof of Lemma I that $\Omega(H, H^0) \geq 1$ for all $H \leq \underline{H}$. Therefore, for all $H \leq \underline{H}$,

$$\Omega(H, H^{0\prime}) > \Omega(H, H^0) \geq 1.$$

It follows that, when the outside option is $H^{0\prime}$, there is no equilibrium aggregator level weakly below $\underline{H}$. Therefore, $\underline{H} < \underline{H}'$. The fact that $\overline{H} < \overline{H}'$ follows from the same line of argument. This establishes point (iii) in the proposition.

Points (i), (ii) and (iv) follow from the fact that a firm's profit is equal to its ι-markup minus one (Theorem 1), m^f is decreasing (Lemma H), and r_j is increasing (Lemma D). □

IX.3 Comparative Statics with Respect to Marginal Costs

The goal of this section is to construct a discrete/continuous choice model $(h_j)_{j \in \mathcal{N}}$ and a firm partition $\mathcal{F}$ such that: (a) Pricing game $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$ has a unique equilibrium for every $(c_j)_{j \in \mathcal{N}}$; (b) There exists a marginal cost vector $(c_j)_{j \in \mathcal{N}}$ and a product j such that, starting from $(c_j)_{j \in \mathcal{N}}$, a small increase in c_j *raises* the equilibrium aggregator level.

Fix an arbitrary pricing game $((h_j)_{j \in \mathcal{N}}, \mathcal{F}, (c_j)_{j \in \mathcal{N}})$. We start by deriving a necessary and sufficient condition under which the aggregate fitting-in function shifts upward (locally) after an increase in c_j $(j \in f)$.[8] In the following, we make explicit the dependence of function m^f on c_j by writing $m^f(H, c_j)$. We also write $r_k(\mu^f, c_k)$ for every k. Differentiating equation (7) with respect to c_j and μ^f, and using equation (7) to eliminate H, we obtain:

$$\frac{\partial m^f}{\partial c_j} = -\frac{m^f(m^f-1)(-\gamma_j')\frac{\partial r_j}{\partial c_j}}{\sum_{k\in f}\left(\gamma_k + m^f(m^f-1)\frac{\partial r_k}{\partial \mu^f}(-\gamma_k')\right)}.$$

It is straightforward to check that $\partial r_j/\partial c_j > 0$. Therefore, $\partial m^f/\partial c_j < 0$.

Next, let $H^f(H, c_j) \equiv \sum_{k\in f} h_k(r_k(m^f(H, c_j), c_k))$ be firm f's contribution to the aggregator. Note that an infinitesimal increase in c_j implies a local upward shift in the aggregate fitting-in function if and only if $\partial H^f/\partial c_j > 0$. Let $\xi = \sum_{k\in f}\left(\gamma_k + m^f(m^f-1)\frac{\partial r_k}{\partial \mu^f}(-\gamma_k')\right)$, and, as in Section VII.1, $\omega^f = (\mu^f - 1)/\mu^f$. Recall that $\frac{\partial r_k}{\partial \mu^f} = \frac{\gamma_k}{(-\gamma_k')\mu^f(1-\omega^f\theta_k)}$ (see the proof of Lemma VIII). Then,

$$\begin{aligned}
\frac{\partial H^f}{\partial c_j} &= \frac{\partial r_j}{\partial c_j}h_j' + \frac{\partial m^f}{\partial c_j}\sum_{k\in f}\frac{\partial r_k}{\partial \mu^f}h_k', \\
&= \frac{1}{\xi}\frac{\partial r_j}{\partial c_j}\left(-(-h_j')\xi + m^f(m^f-1)(-\gamma_j')\sum_{k\in f}\frac{\partial r_k}{\partial \mu^f}(-h_k')\right), \\
&= \frac{1}{\xi}\frac{\partial r_j}{\partial c_j}\sum_{k\in f}\left(-(-h_j')\left(\gamma_k + m^f(m^f-1)\frac{\partial r_k}{\partial \mu^f}(-\gamma_k')\right) + (-\gamma_j')m^f(m^f-1)\frac{\partial r_k}{\partial \mu^f}(-h_k')\right), \\
&= \frac{-\gamma_j'}{\xi}\frac{\partial r_j}{\partial c_j}\sum_{k\in f}\gamma_k\left(-\theta_j\left(1+\frac{m^f-1}{1-\omega^f\theta_k}\right) + \frac{(m^f-1)\theta_k}{1-\omega^f\theta_k}\right), \\
&= \frac{-\gamma_j'}{\xi}\frac{\partial r_j}{\partial c_j}\sum_{k\in f}\gamma_k\left(-\theta_j + \frac{\omega^f}{1-\omega^f}\frac{\theta_k-\theta_j}{1-\omega^f\theta_k}\right).
\end{aligned}$$

If $f = \{1, 2\}$ and $j = 1$, then $\partial H^f/\partial c_1 > 0$ if and only if

$$-\gamma_1\theta_1 + \gamma_2\left(-\theta_1 + \frac{\omega^f}{1-\omega^f}\frac{\theta_2-\theta_1}{1-\omega^f\theta_2}\right) > 0, \tag{xl}$$

where $\omega^f = \frac{m^f(H,c_1)-1}{m^f(H,c_1)}$, functions γ_1 and θ_1 are evaluated at price $p_1 = r_1(m^f(H, c_1), c_1)$, and functions γ_2 and θ_2 are evaluated at price $p_2 = r_2(m^f(H, c_1), c_2)$.

The next step is to find a product pair $(h_1, h_2) \in (\mathcal{H}^\iota)^2$, a marginal cost pair (c_1, c_2), and an aggregator level $H^* > 0$ such that firm f satisfies condition (b) in Theorem 2, and condition (xl) holds. Let product h_2 be a CES product with quality a_2 and $\sigma = 2$: $h(p_2) = a_2/p_2$. Let $h_1(p_1) = 1/\log(1 + e^{p_1})$. Routine calculations show that $h_1 \in \mathcal{H}^\iota$,

[8] To simplify the exposition, we assume that firm f sets finite prices for all its products. This condition holds in the example we construct below.

$\bar{\mu}_1 = \bar{\mu}_2 = 2$, and ρ_1 is strictly increasing. Therefore, firm $f = \{1, 2\}$ satisfies condition (b) in Theorem 2. Moreover, using the properties of CES products ($\theta_2 = 2$) allows us to simplify condition (xl) as follows:

$$-\gamma_1\theta_1 + \gamma_2 \left(-\theta_1 + \frac{\omega^f}{1-\omega^f} \frac{2-\theta_1}{1-2\omega^f} \right) > 0, \tag{xli}$$

Fix $c_2 > 0$ at some arbitrary value. We need to find $H^* > 0$, $a_2 > 0$ and $c_1 > 0$ such that condition (xl) holds.

Let $\mu^f \in (1, 2)$ and $\omega^f = (\mu^f - 1)/\mu^f$. Note that, as c_1 tends to zero, $r_1(\mu^f, c_1)$ converges to a strictly positive real $p_1 = r_1(0, \mu^f)$, which is the unique solution of equation $\iota_1(p_1) = \mu^f$, or, equivalently, $\chi_1(p_1) = \omega^f$. At the limit, the term in parentheses in equation (xli) can then be rewritten as follows:

$$\psi(p_1) = -\theta_1(p_1) + \frac{\chi_1(p_1)}{1-\chi_1(p_1)} \frac{2-\theta_1(p_1)}{1-2\chi_1(p_1)}.$$

Studying function ψ, we show that $\psi(p_1) > 0$ (and $\iota_1(p_1) > 1$) for p_1 high enough. Fix such a p_1, and let $\mu^f \equiv \iota_1(p_1)$ and $\omega^f = (\mu^f - 1)/\mu^f$. Then, by definition of p_1,

$$-\theta_1(r_1(\mu^f, 0)) + \frac{\omega^f}{1-\omega^f} \frac{2-\theta_1(r_1(\mu^f, 0))}{1-2\omega^f} > 0.$$

Therefore, by continuity,

$$-\theta_1(r_1(\mu^f, c_1)) + \frac{\omega^f}{1-\omega^f} \frac{2-\theta_1(r_1(\mu^f, c_1))}{1-2\omega^f} > 0$$

for $c_1 > 0$ small enough. Fix such a c_1.

Let us now inspect the expression in the left-hand side of condition (xli) (recall that, since good 2 is a CES product with $\sigma = 2$, $\gamma_2 = h_2/2$):

$$-\gamma_1(r_1(\mu^f, c_1))\theta_1(r_1(\mu^f, c_1)) + \frac{1}{2} \frac{a_2}{r_2(\mu^f, c_2)} \left(-\theta_1(r_1(\mu^f, c_1)) + \frac{\omega^f}{1-\omega^f} \frac{2-\theta_1(r_1(\mu^f, c_1))}{1-2\omega^f} \right).$$

Clearly, the above expression is strictly positive for high enough a_2. Fix such an a_2. Recall that $m^f(\cdot, c_1)$ is continuous, and decreases from $\bar{\mu}^f (= 2)$ to 1 as H increases from 0 to ∞ (Lemma H). Therefore, there exists $H^* > 0$ such that $m^f(H^*, c_1) = \mu^f$. This concludes the second step of our construction.

The last step is to construct a second firm, firm g, such that the pricing game between firms f and g gives rise to a unique equilibrium, and the equilibrium aggregator level is H^*. Before constructing firm g, we state and prove the following lemma:

Lemma XXII. *Let $(h_j)_{j \in \mathcal{N}} \in (H^\iota)^{\mathcal{N}}$ such that $\bar{\mu}_j = \bar{\mu} < \infty$ and ρ_j is non-decreasing* *for every $j \in \mathcal{N}$, and $(c_j)_{j\in\mathcal{N}} \in \mathbb{R}^{\mathcal{N}}_{++}$. Suppose that a monopolist owns all the products in $\mathcal{N}$, and that consumers have access to an outside option $H^0 > 0$. Then, the monopolist's profit-maximization problem has a unique solution. The aggregator level at the monopolist's optimum, $\hat{H}(H^0)$, is a strictly increasing function of H^0. Moreover, $\lim_0 \hat{H} = 0$, and $\lim_\infty \hat{H} = \infty$.*

Proof. We know from Lemma G that the monopoly problem has a unique solution for every $H^0 > 0$. Therefore, function $\hat{H}(\cdot)$ is well defined. The monopolist's optimal ι-markup, denoted $\hat{\mu}(H^0) \in (1, \bar{\mu}^f)$, is the unique solution of equation (20). It is straightforward to show, e.g., by applying the implicit function theorem to equation (20), that $\hat{\mu}$ is continuous and strictly decreasing. It follows that

$$\hat{H}(H^0) = H^0 + \sum_{j\in\mathcal{N}} h_j(r_j(\hat{\mu}(H^0)))$$

is strictly increasing in H^0. The monopolist earns $\hat{\mu}(H^0) - 1$ at its optimum. Let $m(\cdot)$ be the monopolist's fitting-in function. Then, by definition of m, $m(\hat{H}(H^0)) = \hat{\mu}(H^0)$.

Clearly, $\lim_\infty \hat{H} = \infty$. By monotonicity, $\underline{H} = \lim_0 \hat{H}$ exists, and is non-negative. Assume for a contradiction that $\underline{H} > 0$. Then, for every $H^0 > 0$,

$$\hat{\mu}(H^0) = m(\hat{H}(H^0)) < m(\underline{H}) < \bar{\mu}.$$

For every $\mu \in (1, \bar{\mu})$ and $H^0 > 0$, let $\pi(\mu, H^0)$ be the monopolist's profit when it sets ι-markup μ, and the value of the outside option is H^0. Note that, for every $H^0 > 0$ and $\mu \in (1, \bar{\mu})$,

$$\pi(\mu, H^0) \leq \hat{\mu}(H^0) - 1 \leq m(\underline{H}) - 1.$$

Therefore,

$$\bar{\pi} \equiv \sup_{H^0>0,\ \mu\in(1,\bar{\mu})} \pi(\mu, H^0) \leq m(\underline{H}) - 1 < \bar{\mu} - 1.$$

Moreover, using the definition of ι-markup μ and function γ_j ($j \in \mathcal{N}$), we can rewrite $\pi(\mu, H^0)$ as follows:

$$\pi(\mu, H^0) = \mu \frac{\sum_{j\in\mathcal{N}} \gamma_j(r_j(\mu))}{H^0 + \sum_{j\in\mathcal{N}} h_j(r_j(\mu))}.$$

Note that, for every $\mu \in (1, \bar{\mu})$,

$$\bar{\pi} \geq \mu \frac{\sum_{j\in\mathcal{N}} \gamma_j(r_j(\mu))}{\sum_{j\in\mathcal{N}} h_j(r_j(\mu))} = \mu \frac{\sum_{j\in\mathcal{N}} \gamma_j(r_j(\mu))}{\sum_{j\in\mathcal{N}} \rho_j(r_j(\mu))\gamma_j(r_j(\mu))} \geq \mu \frac{\bar{\mu}-1}{\bar{\mu}} \frac{\sum_{j\in\mathcal{N}} \gamma_j(r_j(\mu))}{\sum_{j\in\mathcal{N}} \gamma_j(r_j(\mu))} = \mu \frac{\bar{\mu}-1}{\bar{\mu}},$$

where the second inequality comes from the fact that, for every j, ρ_j is non-decreasing and $\lim_\infty \rho_j = \bar{\mu}/(\bar{\mu}-1)$ by Lemma III-(f). Taking the limit as μ tends to $\bar{\mu}$ allows us to conclude that $\bar{\pi} \geq \bar{\mu} - 1$, which is a contradiction. □

Firm f satisfies all the assumptions in Lemma XXII. Therefore, function $\hat{H}(\cdot)$ is a bijection from $(0,\infty)$ to $(0,\infty)$, and there exists a unique $H^0 > 0$ such that $\hat{H}(H^0) = H^*$. By definition of $\hat{H}$, this means that

$$H^* = H^0 + \sum_{k\in f} h_k(r_k(m^f(H^*, c_1), c_k)).$$

Next, we construct a firm g such that, when the aggregator level is H^*, firm g's contribution to the aggregator is H^0. To do so, we rely on the results derived in Section 5.1. Let g be an arbitrary multiproduct firm selling only CES products (with a common σ). Denote firm g's type by $T^g > 0$. We know that, when the aggregator level is H^*, firm g's contribution to the aggregator is $S(T^g/H^*)H^*$. Moreover, $S(\cdot)$ is continuous and strictly increasing, and it is straightforward to show that $\lim_0 S = 0$ and $\lim_\infty S = 1$. Therefore, there exists a unique $\hat{T}^g$ such that $S(\hat{T}^g/H^*)H^* = H^0$.

We can conclude. We have constructed a multiproduct duopoly pricing game with two firms, f and g. By construction, firm f satisfies condition (b) in Theorem 2. Since firm g only sells CES products with a common σ, firm g satisfies condition (a) in Theorem 2. Therefore, the pricing game between firms f and g has a unique equilibrium for every marginal cost vector for firm f and for every value of T^g. When firm f's marginal costs are equal to c_1 and c_2, as defined above, and firm g's type is $\hat{T}^g$, the equilibrium aggregator level is H^*. An infinitesimal increase in the value of c_1 induces a local upward shift in the aggregate fitting-in function. Since that function is has a finite limit when $H \to \infty$ and has a unique fixed point, it follows that the equilibrium value of the aggregator increases. Therefore, consumer surplus increases, and both firms' profits decrease.

X Applications: Merger Analysis and International Trade

X.1 Static Merger Analysis: Proof of Proposition 12

Proof. Let

$$\hat{T}^M \equiv H^* S^{-1}\left(\sum_{f\in\mathcal{I}} S\left(\frac{T^f}{H^*}\right)\right). \tag{xlii}$$

$\hat{T}^M$ is well-defined, since S is strictly increasing and has range $(0,1)$.

If $T^M = \hat{T}^M$, we have:

$$1 = \sum_{f\in\mathcal{F}} S\left(\frac{T^f}{H^*}\right) = S\left(\frac{T^M}{H^*}\right) + \sum_{f\in\mathcal{O}} S\left(\frac{T^f}{H^*}\right),$$

where the first equality is the pre-merger equilibrium condition whereas the second equality follows from $T^M = \hat{T}^M$. Therefore, $\hat{H}^* = H^*$, i.e., the merger is CS-neutral if $T^M = \hat{T}^M$. As

$S'(\cdot) > 0$, if $T^M > \hat{T}^M$, we have

$$S\left(\frac{T^M}{H^*}\right) + \sum_{f \in \mathcal{O}} S\left(\frac{T^f}{H^*}\right) > 1,$$

implying that $\hat{H}^* > H^*$, so the merger is CS-increasing. Similarly, if $T^M < \hat{T}^M$, then $\hat{H}^* < H^*$, so the merger is CS-decreasing.

Next, we note that a CS-neutral merger involves synergies in that $\hat{T}^M > \sum_{f \in \mathcal{I}} T^f$. Suppose otherwise that $\hat{T}^M \leq \sum_{f \in \mathcal{I}} T^f$. Then,

$$S\left(\frac{\hat{T}^M}{H^*}\right) \leq S\left(\sum_{f \in \mathcal{I}} \frac{T^f}{H^*}\right) < \sum_{f \in \mathcal{I}} S\left(\frac{T^f}{H^*}\right),$$

where the first inequality follows from $S'(\cdot) > 0$ and the second from Lemma XXI. But then the merger would be CS-decreasing, a contradiction. Hence, $\hat{T}^M > \sum_{f \in \mathcal{I}} T^f$.

We now show that a CS-neutral merger is profitable. Recall that, under CES demands, $\pi = m - 1$ and $S = \frac{\sigma}{\sigma - 1} \frac{m-1}{m}$. It follows that $\pi = \frac{\sigma - 1}{\sigma} mS$. Similarly, under MNL demands, $\pi = mS$. In both cases, π is proportional to mS. Note that

$$m\left(\frac{T^M}{H^*}\right) S\left(\frac{T^M}{H^*}\right) = m\left(\frac{T^M}{H^*}\right) \sum_{f \in \mathcal{I}} S\left(\frac{T^f}{H^*}\right) > \sum_{f \in \mathcal{I}} m\left(\frac{T^f}{H^*}\right) S\left(\frac{T^f}{H^*}\right),$$

where the equality follows because the merger is CS-neutral, and the inequality follows because $\hat{T}^M > T^f$ for every $f \in \mathcal{I}$ and $m'(\cdot) > 0$. Hence, merger M is profitable if $T^M = \hat{T}^M$. Next, suppose that the merger is CS-increasing, i.e., $T^M > \hat{T}^M$. Then, by Proposition 11-(ii), the merged firm makes a strictly higher equilibrium profit than when its type is $\hat{T}^M$. This implies in particular that the merger is profitable.

Finally, we establish the existence of threshold $\tilde{T}^M$. Note first that, if $T^M = \hat{T}^M$, then the merger is W-increasing, since it raises the joint profits of the merging parties, but affects neither consumer surplus, nor the outsiders' profits. On the other hand, it is straightforward to show that, as T^M tends to zero, $\bar{H}^*$ converges to the equilibrium aggregator level which would prevail if only the outsiders were present. Social welfare in that case is equal to the limit of social welfare pre-merger as T^f tends to 0 for every $f \in \mathcal{I}$, which, by monotonicity, is strictly lower than equilibrium social welfare when $T^f > 0$ for every $f \in \mathcal{I}$. Therefore, the merger is W-decreasing if T^M is low enough. By the intermediate value theorem, there exists $\tilde{T}^M$ such that the welfare is W-neutral if $T^M = \tilde{T}^M$. By monotonicity of social welfare, the merger is W-increasing if $T^M > \tilde{T}^M$, and W-decreasing if $T^M < \tilde{T}^M$. $\square$

X.2 Static Merger Analysis: External Effects

We first derive formulas for η:

Lemma XXIII. *$\eta(H)$ is given by:*

$$\eta(H) = -1 + \sum_{f \in \mathcal{O}} \phi_\alpha(s^f),$$

where $\alpha = (\sigma - 1)/\sigma$ in the CES case, $\alpha = 1$ in the MNL case, $s^f = S(T^f/H)$, and

$$\phi_\alpha(s) = \frac{\alpha s(1-s)}{(1-\alpha s)(1-s+\alpha s^2)}, \quad \forall s \in (0,1).$$

Proof. This follows from the definition of η and from the fact that

$$\begin{aligned}
xm'(x) &= x\alpha \frac{S'(x)}{(1-\alpha S(x))^2}, \text{ since } m(x) = \frac{1}{1-\alpha S(x)}, \\
&= \frac{\alpha}{(1-\alpha S(x))^2} \frac{S(x)(1-S(x))(1-\alpha S(x))}{1-S(x)+\alpha S(x)^2}, \text{ using equation (xxxvii)}, \\
&= \frac{\alpha S(x)(1-S(x))}{(1-\alpha S(x))(1-S(x)+\alpha S(x)^2)}, \\
&= \phi_\alpha(S(x)).
\end{aligned}$$

□

Next, we study the properties of function $\phi_\alpha(s)$:

Lemma XXIV. *Function $(s, \alpha) \mapsto \phi_\alpha(s)$ has the following properties:*

(a) For every $s \in (0,1)$, $\alpha \mapsto \phi_\alpha(s)$ is strictly increasing.

There exists $\hat{\alpha} \in (0,1)$ such that:

(b) If $\alpha \leq \hat{\alpha}$, then $\phi_\alpha(s) \leq s$ for every $s \in [0,1]$.

(c) If $\alpha > \hat{\alpha}$, then there exist $0 \leq \underline{s}(\alpha) < \overline{s}(\alpha) \leq 1$ such that, for every $s \in [0,1]$, $\phi_\alpha(s) > s$ if and only if $s \in (\underline{s}(\alpha), \overline{s}(\alpha))$.

Moreover, if $\alpha > \hat{\alpha}$, then there exist thresholds $s^*(\alpha) \in (0,1]$ and $\hat{s}(\alpha) \in (0,1)$ such that:[9]

(d) $s \mapsto \phi_\alpha(s)$ is strictly increasing on $(0, s^*(\alpha))$ and strictly decreasing on $(s^*(\alpha), 1)$.

(e) $s \mapsto \phi_\alpha(s)$ is strictly convex on $(0, \hat{s}(\alpha))$ and strictly concave on $(\hat{s}(\alpha), 1)$.

Proof. We prove the lemma (analytically) using Mathematica. Mathematica files are available upon request. □

[9] More on thresholds $\underline{s}(\alpha)$, $\overline{s}(\alpha)$, $s^*(\alpha)$ and $\hat{s}(\alpha)$:

- In the MNL case, $\underline{s}(1) = 0$ and $\overline{s}(1) = 1$. Otherwise, both thresholds are interior.
- In the MNL case, $s^*(1) = 1$. Otherwise, $0.68 \leq s^*(\alpha) < 1$.
- $0.28 \leq \hat{s}(\alpha) < 1$.

The following lemma is the final step toward proving proposition 13:

Lemma XXV. *Let* $\bar{\alpha} = \frac{3}{2}(\sqrt{57} - 7) \simeq 0.82$*. If* $\alpha \leq \bar{\alpha}$*, then any infinitesimal CS-decreasing merger has a negative external effect. If instead* $\alpha > \bar{\alpha}$*, then there exist infinitesimal CS-decreasing mergers that have positive external effects, and infinitesimal CS-increasing mergers that have negative external effects.*

Proof. Define

$$\mathcal{S} = \bigcup_{n \geq 1} \mathcal{S}^n, \text{ where } \mathcal{S}^n = \{s \in [0,1]^n : \sum_{i=1}^{n} s_i \leq 1\} \ \forall n \geq 1,$$

$$\bar{\mathcal{S}} = \bigcup_{n \geq 1} \mathcal{S}^n, \text{ where } \bar{\mathcal{S}}^n = \{s \in [0,1]^n : \sum_{i=1}^{n} s_i = 1\} \ \forall n \geq 1,$$

and[10]

$$\Psi(\alpha) = \sup_{s \in \mathcal{S}} \sum_i \phi_\alpha(s_i), \ \forall \alpha \in (\hat{\alpha}, 1].$$

Clearly, since $\phi_\alpha(s) \geq 0$ for all s, we have that $\Psi(\alpha) = \sup_{s \in \bar{\mathcal{S}}} \sum_i \phi_\alpha(s_i)$. Next, we claim that $\Psi(\alpha) = \sup_{s \in \bar{\mathcal{S}}^4} \sum_{i=1}^4 \phi_\alpha(s_i)$. To prove this, we show that, for every $s \in \bar{\mathcal{S}}$, there exists $s' \in \bar{\mathcal{S}}^4$ such that

$$\sum_i \phi_\alpha(s_i) \leq \sum_{i=1}^{4} \phi_\alpha(s'_i).$$

If s belongs to $\mathcal{S}^n$ for some $n \leq 4$, or, more generally, if s has at most four components different from zero, then this is obvious. Assume instead that s has five or more components different from zero. Assume without loss of generality that $s \in \bar{\mathcal{S}}^n$ for some $n \geq 5$, that $s_i > 0$ for every i, and that the components of s_i have been sorted in increasing order. We construct s' by induction.

Let us first define a function ξ, which takes as argument a profile of market shares $s \in \bar{\mathcal{S}}^n$ sorted in increasing order and with strictly positive components, and returns a profile of market shares $\xi(s)$ sorted in increasing order and with strictly positive components, such that either $\xi(s) \in \bar{\mathcal{S}}^n$, or $\xi(s) \in \overline{\mathcal{S}^{n-1}}$. ξ is defined as follows:

- If $s_2 \geq \hat{s}(\alpha)$ (or if $s \in \mathcal{S}^1$), then $\xi(s) = s$.
- If $s_2 < \hat{s}(\alpha)$, then do the following:
  - If $s_1 + s_2 \leq \hat{s}(\alpha)$, then form the $(n-1)$-dimensional vector with first component $s_1 + s_2$ and remaining components $(s_i)_{3 \leq i \leq n}$, and sort that vector in increasing order to obtain $\xi(s)$.

[10]Notation: Let $s \in \mathcal{S}$ and $n \geq 1$ such that $s \in \mathcal{S}^n$. We write

$$\sum_i \phi_\alpha(s_i) = \sum_{i=1}^{n} \phi_\alpha(s_i).$$

– If instead $s_1 + s_2 > \hat{s}(\alpha)$, then form the n-dimensional with first component $s_1 + s_2 - \hat{s}(\alpha)$, second component $\hat{s}(\alpha)$, and remaining components $(s_i)_{3\leq i\leq n}$, and sort that vector in increasing order to obtain $\xi(s)$.

Note that, since $\phi_\alpha(\cdot)$ is convex on $[0, \hat{s}(\alpha)]$, we have that, for every $\tilde{s} \in \bar{\mathcal{S}}$

$$\sum_i \phi_\alpha(\tilde{s}_i) \leq \sum_i \phi_\alpha\left((\xi(\tilde{s}))_i\right)$$

We can now define sequence $(s^k)_{k\geq 0}$ by induction: $s^0 = s$; $s^{k+1} = \xi(s^k)$ for every $k \geq 0$. Let m^k denote the number of components of s^k greater or equal to $\hat{s}(\alpha)$, and n^k denote the dimensionality of vector s^k. By definition of ξ and of sequence $(s^k)_{k\geq 0}$, sequence of integers $(m^k)_{k\geq 0}$ (resp. $(n^k)_{k\geq 0}$) is non-decreasing (resp. non-increasing) and bounded above by n (resp. bounded below by 1). Therefore, those sequences of integers are eventually stationary: There exists $K \geq 0$ such that $m^k = m^{k+1}$ and $n^k = n^{k+1}$ for every $k \geq K$. It follows that $(s^k)_{k\geq 0}$ is also stationary after K. Let s' be the stationary value of sequence $(s^k)_{k\geq 0}$. Then, by induction on k,

$$\sum_i \phi_\alpha(s_i) \leq \sum_i \phi_\alpha(s'_i).$$

Moreover, s' has at most one component in $[0, \hat{s}(\alpha))$ (for otherwise, $\xi(s')$ would not be equal to s'). Let n' be the dimensionality of vector s'. Then,

$$1 = \sum_{i=1}^{n'} s'_i \geq (n'-1)\hat{s}(\alpha) \geq 0.28 \times (n'-1),$$

where the last inequality follows by Lemma XXIV (see footnote 9). It follows that $n' \leq 4$. Having constructed s', we can conclude that

$$\Psi(\alpha) = \sup_{s\in\bar{\mathcal{S}}^4} \sum_{i=1}^{n} \phi_\alpha(s_i). \tag{xliii}$$

By continuity of ϕ_α and by compactness of $\bar{\mathcal{S}}^4$, the maximization problem defined by (xliii) has a solution. Let s be such a solution. Then, by the convexity argument used in the construction of s', s has a most one component in $(0, \hat{s}(\alpha))$. Moreover, since ϕ_α is strictly concave on $[\hat{s}(\alpha), 1]$, the components of s that are greater or equal to $\hat{s}(\alpha)$ must be equal to each other. It follows that

$$\Psi(\alpha) = \max_{x\in[0,1]} \max\left(\phi_\alpha(x) + \phi_\alpha(1-x), \phi_\alpha(x) + 2\phi_\alpha\left(\frac{1-x}{2}\right), \phi_\alpha(x) + 3\phi_\alpha\left(\frac{1-x}{3}\right)\right).$$

We (analytically) solve the above maximization problem using Mathematica. We obtain:

$$\Psi(\alpha) = \begin{cases} \frac{18\alpha}{18-3\alpha-\alpha^2} & \text{if } \alpha \leq \frac{6}{7}, \\ \frac{4\alpha}{4-\alpha^2} & \text{otherwise.} \end{cases}$$

It is straightforward to check that Ψ is strictly increasing, and that $\Psi(\hat{\alpha}) < 1 < \Psi(1)$. The unique solution of equation $\Psi(\alpha) = 1$ on interval $(\hat{\alpha}, 1]$ is $\bar{\alpha} = \frac{3}{2}(\sqrt{57} - 7)$.

We can conclude. Assume first that $\alpha \leq \hat{\alpha}$. Then, by Lemma XXIV-(b), $\sum_{f \in \mathcal{O}} \phi_\alpha(s^f) \leq \sum_{f \in \mathcal{O}} s^f < 1$ for every profile of outsiders' market shares $(s^f)_{f \in \mathcal{O}}$. Therefore, any infinitesimal CS-decreasing merger must have a negative external effect.

Next, assume that $\alpha \in (\hat{\alpha}, \bar{\alpha}]$. Then, for every profile of outsiders' market shares $(s^f)_{f \in \mathcal{O}}$,

$$\sum_{f \in \mathcal{O}} \phi_\alpha(s^f) < \phi_\alpha(1 - s^f) + \sum_{f \in \mathcal{O}} \phi_\alpha(s^f) \leq \Psi(\alpha) \leq \Psi(\bar{\alpha}) = 1.$$

Therefore, any infinitesimal CS-decreasing merger must have a negative external effect.

Finally, assume $\alpha > \bar{\alpha}$. We first show that there exists an infinitesimal CS-decreasing merger that has a negative external effect. Let $\mathcal{O} = \{1\}$ and $\mathcal{I} = \{2, 3\}$. Since $\phi_\alpha(\cdot)$ is continuous and $\phi_\alpha(0) = 0$, there exists $s \in (0, 1)$ such that $\phi_\alpha(s) < 1$. Let $T^1 = S^{-1}(s)$, and $T^2 = T^3 = S^{-1}\left(\frac{1-s}{2}\right)$. Then, by construction, the pre-merger equilibrium aggregator level is $H = 1$, and market shares are as follows: $s^1 = s$, $s^2 = s^3 = \frac{1-s}{2}$. The external effect of an infinitesimal and CS-decreasing merger between firms 2 and 3 is given by $\phi_\alpha(s) - 1$, which is strictly negative by construction. Next, we claim that there exists an infinitesimal CS-decreasing merger that has a positive external effect. Since $\Psi(\alpha) > 1$, there exists $(s_i)_{1 \leq i \leq n} \in (0, 1]^n$ such that $\sum_{i=1}^n s_i \leq 1$ and $\sum_{i=1}^n \phi_\alpha(s_i) > 1$. By continuity, for $\varepsilon > 0$ small enough, $\sum_{i=1}^n \phi_\alpha(s_i - \varepsilon) > 1$. Let $\mathcal{O} = \{1, \ldots, n\}$, $\mathcal{I} = \{n+1, n+2\}$, $s'^i = s_i - \varepsilon$ for every $i \in \mathcal{O}$, $s'^i = \frac{1}{2}\left(1 - \sum_{j=1}^n s'^j\right)$ for $i \in \mathcal{I}$, and $T^i = S^{-1}(s'^i)$ for every $i \in \mathcal{I} \cup \mathcal{O}$. Then, by construction, an infinitesimal and CS-decreasing merger between the insiders has a positive external effect.

Since any CS-decreasing merger can be decomposed into the integral of infinitesimal CS-decreasing mergers, and since a CS-decreasing merger can be made infinitesimal by tweaking the post-merger type of the merged entity, the above results extend immediately to non-infinitesimal mergers: If $\alpha \leq \bar{\alpha}$, then any CS-decreasing merger has a negative external effect; If $\alpha > \bar{\alpha}$, then there exist CS-decreasing mergers that have positive external effects, and CS-decreasing mergers that have negative external effects. □

Proposition 13 follows immediately from Lemmas XXIV and XXV.

Finally, we formalize and prove our statements on the impact of the concentration of outsiders' market shares. Fix $\alpha > \bar{\alpha}$. Assume without loss of generality that $\mathcal{O} = \{1, \ldots, n\}$ with $n \geq 2$. An outsider industry structure is a vector of outsiders' market shares $\mathbf{s} \in$

$[0, \hat{s}(\alpha)]^n$ such that $\sum_{i=1}^n s_i < 1$. To every outsider industry structure $\mathbf{s}$, we associate a discrete probability distribution $P_{\mathbf{s}}(\cdot)$, which is defined as follows:

$$P_{\mathbf{s}}(x) = \frac{1}{n} \left| \{ i \in \{1, \ldots, n\} : \quad s_i = x \} \right|, \quad \forall x \in \mathbb{R}.$$

Note that the mean of probability distribution $P_{\mathbf{s}}$ is equal to $\frac{\sum_{i=1}^n s_i}{n}$.

We now use these associated probability distributions to define a partial order on the set of outsider industry structures. We say that outsider industry structure $\mathbf{s}'$ is more concentrated than outsider industry structure $\mathbf{s}$ if $P_{\mathbf{s}}$ and $P_{\mathbf{s}'}$ have the same mean (i.e., the aggregate market shares of the outsiders are the same in both industry structures) and $P_{\mathbf{s}}$ second-order stochastically dominates $P_{\mathbf{s}'}$. For instance, with $n = 2$, industry structure $(0.05, 0.15)$ is more concentrated than industry structure $(0.1, 0.1)$.

Suppose that outsider industry structure $\mathbf{s}'$ is more concentrated than outsider industry structure $\mathbf{s}$. Then, since ϕ_α is convex on a set which contains the supports of $P_{\mathbf{s}}(x)$ and $P_{\mathbf{s}'}(x)$,

$$\int_{\mathbb{R}} \phi_\alpha(x) dP_{\mathbf{s}'}(x) \geq \int_{\mathbb{R}} \phi_\alpha(x) dP_{\mathbf{s}}(x).$$

Using the definition of $P_{\mathbf{s}}$ and $P_{\mathbf{s}'}$, we obtain:

$$\sum_{i=1}^n \frac{1}{n} \phi_\alpha(s_i') \geq \sum_{i=1}^n \frac{1}{n} \phi_\alpha(s_i).$$

Therefore, η is higher with outsider industry structure $\mathbf{s}'$ than with outsider industry structure $\mathbf{s}$. This implies that the external effect of an infinitesimal CS-decreasing merger is more likely to be positive when the outsiders have more concentrated market shares. Note that, by convexity of function $x \mapsto x^2$, the industry HHI is higher under industry structure $\mathbf{s}'$ than under industry structure $\mathbf{s}$.

X.3 Dynamic Merger Analysis: Proof of Proposition 14

Proof. Let $\mathcal{I}$ be the set of insiders associated with merger M. Differentiating equation (xlii), we obtain

$$\begin{aligned}
S'\left(\frac{\hat{T}^M}{H^*}\right) \frac{d\hat{T}^M}{dH^*} &= \frac{\hat{T}^M}{H^*} S'\left(\frac{\hat{T}^M}{H^*}\right) - \sum_{f \in \mathcal{I}} \frac{T^f}{H^*} S'\left(\frac{T^f}{H^*}\right), \\
&= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right) S\left(\frac{\hat{T}^M}{H^*}\right) - \sum_{f \in \mathcal{I}} \varepsilon\left(\frac{T^f}{H^*}\right) S\left(\frac{T^f}{H^*}\right), \\
&= \varepsilon\left(\frac{\hat{T}^M}{H^*}\right) \sum_{f \in \mathcal{I}} S\left(\frac{T^f}{H^*}\right) - \sum_{f \in \mathcal{I}} \varepsilon\left(\frac{T^f}{H^*}\right) S\left(\frac{T^f}{H^*}\right),
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{f \in \mathcal{I}} \left(\varepsilon \left(\frac{\hat{T}^M}{H^*} \right) - \varepsilon \left(\frac{T^f}{H^*} \right) \right) S \left(\frac{T^f}{H^*} \right), \\
&< 0,
\end{aligned}
$$

where the third line follows by definition of $\hat{T}^M$ and the last line follows from Lemma XX and from the fact that $\hat{T}^M > T^f$ for every $f \in \mathcal{I}$. □

X.4 Dynamic Merger Analysis: Dynamic Optimality of Myopic Merger Approval Policy

Consider two mergers, M_1 and M_2, and assume that these mergers are disjoint, i.e., no firm takes part in both.

Proposition V. *If merger M_i is CS-nondecreasing (and hence profitable) in isolation, it remains CS-nondecreasing (and hence profitable) if another merger M_j, $j \neq i$, that is CS-nondecreasing in isolation takes place. If merger M_i is CS-decreasing in isolation, it remains CS-decreasing if another merger M_j, $j \neq i$, that is CS-decreasing in isolation takes place.*

Proof. Suppose M_i is CS-nondecreasing in isolation, which means that $T^{M_1} \geq \hat{T}^{M_1}$. If the CS-nondecreasing merger M_j takes place, the equilibrium value of the aggregator H^* weakly increases, and so – by Proposition 14 – the cutoff $\hat{T}^{M_1}$ weakly decreases. As T^{M_1} was initially above the cutoff, it therefore remains so after M_j has taken place, i.e., M_i is still CS-nondecreasing. A similar argument can be used to show the sign-preserving complementarity for mergers that are CS-decreasing in isolation. The assertion on profitability follows from Proposition 12. □

Proposition VI. *Suppose that merger M_1 is CS-nondecreasing in isolation whereas merger M_2 is CS-decreasing in isolation but CS-nondecreasing once merger M_1 has taken place. Then, merger M_1 is CS-increasing (and hence profitable), conditional on merger M_2 taking place. Moreover, the joint profit of the firms involved in M_1 is strictly larger if both mergers take place than if neither does.*

Proof. As in the proof of Proposition 2 in Nocke and Whinston (2010), reverse the order of the two mergers: Consider first implementing merger M_2 (step 1) and then merger M_1 (step 2). As consumer surplus must, by assumption, be (weakly) higher after both mergers have taken place than before, and because consumer surplus (strictly) falls at step 1 (again, by assumption), consumer surplus must (strictly) increase at step 2. That is, M_1 is CS-increasing, conditional on M_2 taking place. By Proposition 12, this implies that the joint profit of the firms in M_1 must go up at step 2. The joint profit of the firms in M_1 must go up at step 1 as well, as the CS-decreasing merger at step 1 induces a reduction in the equilibrium value of the aggregator, which benefits all outsiders to that merger by Proposition 11-(i). □

We now embed our pricing game in a dynamic model with endogenous mergers and merger policy, as in Nocke and Whinston (2010). There are T periods, and a set $\{M_1, M_2, ..., M_K\}$ of disjoint potential mergers. Merger M_k becomes feasible at the beginning of period t with probability $p_{kt} \in [0, 1]$, where $\sum_t p_{kt} \leq 1$. Conditional on becoming feasible, the post-merger type of the merged firm M_k is drawn from some distribution C_{kt}. The feasibility of a particular merger (including its efficiency) is publicly observed by all firms. In each period, the firms involved in a feasible and not-yet-approved merger decide whether or not to propose their merger to the antitrust authority. Bargaining is efficient so that the merger partners propose the merger if and only if it is in their joint interest to do so. Given a set of proposed mergers, the antitrust authority then decides which mergers to approve (if any). An approved merger is consummated immediately. Finally, at the end of each period, the firms play the pricing game, given current market structure. All firms as well as the antitrust authority discount payoffs with factor $\delta \leq 1$.

Following Nocke and Whinston (2010), we define a *myopically CS-maximizing merger policy* as an approval policy, where in each period, given the set of proposed mergers and current market structure, the antitrust authority approves a set of mergers that maximizes consumer surplus in the current period. The *most lenient myopically CS-maximizing merger policy* is a *myopically CS-maximizing merger policy* that approves the largest such set (i.e., including CS-neutral mergers). (As shown in Nocke and Whinston (2010) such a policy is well-defined.)

The following proposition shows that Nocke and Whinston (2010)'s result on the dynamic optimality of a myopic merger approval policy carries over to our multiproduct firm setting:

Proposition VII. *Suppose the antitrust authority adopts the most lenient myopically CS-maximizing merger policy. Then, all feasible mergers being proposed in each period after any history is a subgame-perfect Nash equilibrium for the firms. The equilibrium outcome maximizes discounted consumer surplus (indirect utility) for any realized sequence of feasible mergers. Moreover, for each such sequence, every subgame-perfect Nash equilibrium results in the same optimal sequence of period-by-period consumer surpluses.*

Proof. The result follows from Propositions 12, 14, V, and VI, which are the analogues of Corollary 1 and Proposition 1 and 2 in Nocke and Whinston (2010). See Nocke and Whinston (2010) for details. □

X.5 Trade Analysis: Results on Productivity, Inter- and Intra-Firm Size Distributions, and Welfare

Inter-firm size distribution

Proposition VIII. *Suppose demand is either of the CES or MNL form. Then, for $T^f > T^g$, the ratio $S(T^f/H)/S(T^g/H)$ is increasing in H. That is, a trade liberalization leads to a smaller fractional decrease in the market share of a larger than a smaller firm.*

Proof. We have

$$\frac{d}{dH}\left(\frac{S\left(\frac{T^f}{H}\right)}{S\left(\frac{T^g}{H}\right)}\right) > 0$$

if and only if

$$\frac{\frac{T^f}{H}S'\left(\frac{T^f}{H}\right)}{S\left(\frac{T^f}{H}\right)} < \frac{\frac{T^g}{H}S'\left(\frac{T^g}{H}\right)}{S\left(\frac{T^g}{H}\right)}.$$

By Lemma XX, $\epsilon'(x) < 0$ for all x, where $\epsilon(x) \equiv xS'(x)/S(s)$. Hence, the inequality holds if and only if $T^f > T^g$. □

Proposition IX. *Suppose demand is of the MNL form. Then, for $T^f > T^g$, the sales ratio between firms f and g is increasing in H if and only if $\phi(s^f, \bar{c}^f) < \phi(s^g, \bar{c}^g)$, where $s^i = S(T^i/H)$ and*

$$\bar{c}^i \equiv \sum_{k\in i}\left(\frac{e^{\frac{a_k-c_k}{\lambda}}}{\sum_{j\in i} e^{\frac{a_j-c_j}{\lambda}}}\right)c_k$$

are, respectively, the market share (in volume) and the (output-weighted) average marginal cost of firm $i \in \{f, g\}$, and

$$\phi(s, \bar{c}) \equiv \frac{1-s}{1-s+s^2}\left(1-s+\frac{s}{1+\frac{\bar{c}(1-s)}{\lambda}}\right)$$

is decreasing in s and $\bar{c}$.

Proof. Firm f's sales can be written as

$$\begin{aligned}
\text{Sales}^f &= \frac{1}{H}\left(\sum_{j\in f} p_j \exp\frac{a_j-p_j}{\lambda}\right), \\
&= \frac{1}{H}\left(\sum_{j\in f}(\lambda\mu^f + c_j)\,\mathrm{e}^{-\mu^f}\exp\frac{a_j-c_j}{\lambda}\right), \\
&= \frac{T^f}{H}\,\mathrm{e}^{-\mu^f}\left(\lambda\mu^f + \bar{c}^f\right).
\end{aligned}$$

The fact that $s^f = \frac{T^f}{H}\mathrm{e}^{-\mu^f}$ and $\mu^f = \frac{1}{1-s^f}$ allows us to rewrite firm f's sales as follows:

$$\text{Sales}^f = s^f\left(\frac{\lambda}{1-s^f} + \bar{c}^f\right).$$

The logarithmic derivative of sales with respect to H is given by:

$$\begin{aligned}\frac{d\log \text{Sales}^f}{dH} &= \frac{ds^f}{dH}\left(\frac{1}{s^f}+\frac{1}{(1-s^f)+\frac{\bar{c}^f(1-s^f)^2}{\lambda}}\right),\\ &= -\frac{T^f}{H^2}S'\left(\frac{T^f}{H}\right)\left(\frac{1}{s^f}+\frac{1}{(1-s^f)+\frac{\bar{c}^f(1-s^f)^2}{\lambda}}\right),\\ &= -\frac{1}{H}\varepsilon\left(\frac{T^f}{H}\right)s^f\left(\frac{1}{s^f}+\frac{1}{(1-s^f)+\frac{\bar{c}^f(1-s^f)^2}{\lambda}}\right),\end{aligned}$$

where ε is the elasticity of S. Recall that (see equation (xxxvii))

$$\varepsilon = \frac{(1-S)^2}{1-S+S^2}.$$

It follows that

$$\frac{d\log \text{Sales}^f}{dH} = -\frac{1}{H}\underbrace{\frac{1-s^f}{1-s^f+s^{f2}}\left(1-s^f+\frac{s^f}{1+\frac{\bar{c}^f(1-s^f)}{\lambda}}\right)}_{\equiv\phi(s^f,\bar{c}^f)}$$

and hence,

$$\frac{d\log(\text{Sales}^f/\text{Sales}^g)}{dH} = \frac{1}{H}\left(\phi(s^g,\bar{c}^g)-\phi(s^f,\bar{c}^f)\right).$$

It can be verified that ϕ is decreasing in both arguments. □

Intra-firm size distribution

Proposition X. *Suppose demand is either of the CES or MNL form. Then, for $j, k \in f$, the market share ratio s_j/s_k is independent of H. That is, a trade liberalization leads to the same fractional decrease in the market share of all products offered by the same firm.*

Proof. Consider first the case of CES demand. The ratio of market shares (in value) between any two products $j, k \in f$ is given by

$$\frac{s_j}{s_k} = \frac{a_j}{a_k}\left(\frac{p_j}{p_k}\right)^{1-\sigma} = \frac{a_j}{a_k}\left(\frac{\frac{c_j}{1-\mu^f/\sigma}}{\frac{c_k}{1-\mu^f/\sigma}}\right)^{1-\sigma} = \frac{a_j}{a_k}\left(\frac{c_j}{c_k}\right)^{1-\sigma}.$$

Hence, the market share ratio is independent of H.

Consider now the case of MNL demand. The ratio of market shares (in volume) between any two products $j, k \in f$ is given by

$$\frac{s_j}{s_k} = \frac{e^{\frac{a_j-p_j}{\lambda}}}{e^{\frac{a_k-p_k}{\lambda}}} = \frac{e^{\frac{a_j-c_j}{\lambda}-\mu^f}}{e^{\frac{a_k-c_k}{\lambda}-\mu^f}} = \frac{e^{\frac{a_j-c_j}{\lambda}}}{e^{\frac{a_k-c_k}{\lambda}}},$$

which is independent of H. $\square$

Proposition XI. *Suppose demand is of the MNL form. Then, for $j, k \in f$, with $c_j > c_k$, the sales ratio $(p_j s_j)/(p_k s_k)$ is increasing in H. That is, within each firm, a trade liberalization leads to a larger fractional increase in the sales of a product that is produced at higher marginal cost.*

Proof. The ratio of sales of any two products $j, k \in f$ is given by

$$\frac{p_j s_j}{p_k s_k} = \frac{c_j + \lambda \mu^f}{c_k + \lambda \mu^f} \frac{\mathrm{e}^{\frac{a_j - c_j}{\lambda}}}{\mathrm{e}^{\frac{a_k - c_k}{\lambda}}}.$$

As an increase in H induces a decrease in the markup μ^f, this ratio is increasing in H if and only if $c_j > c_k$. $\square$

Productivity. We argue in the paper that a monotone transformation of firm f's type provides a theoretically sound measure of that firm's productivity. We now prove this assertion formally.

The composite commodity approach. Assume that demand is of the CES form, and let $\alpha = (\sigma - 1)/\sigma$. The composite commodity produced by firm f has been defined as $Q^f = \left(\sum_{j \in f} a_j^{1-\alpha} q_j^{\alpha}\right)^{\frac{1}{\alpha}}$. Suppose that firm f has been tasked to produce a certain level Q^f of composite commodity in a cost-minimizing way. Then, firm f solves the following cost-minimization problem:

$$\min_{(q_j)_{j \in f}} \sum_{j \in f} c_j q_j \quad \text{s.t. } Q^f = \left(\sum_{j \in f} a_j^{1-\alpha} q_j^{\alpha}\right)^{\frac{1}{\alpha}}.$$

The first-order condition for product $i \in f$ is:

$$c_i - \Lambda (Q^f)^{1-\alpha} a_i^{1-\alpha} q_i^{\alpha - 1} = 0,$$

where Λ is the Lagrange multiplier associated with the output constraint. Multiplying the first-order condition by q_i and adding up, we obtain: $\sum_{j \in f} c_j q_j = \Lambda Q^f$.

Moreover,

$$q_i = \left(\frac{\Lambda}{c_i}\right)^{\frac{1}{1-\alpha}} a_i Q^f.$$

Therefore,

$$(Q^f)^{\alpha} = \sum_{i \in f} a_i^{1-\alpha} q_i^{\alpha} = \sum_{i \in f} \left(\frac{\Lambda}{c_i}\right)^{\frac{\alpha}{1-\alpha}} a_i (Q^f)^{\alpha}.$$

It follows that

$$\Lambda^{1-\sigma} = \Lambda^{-\frac{\alpha}{1-\alpha}} = \sum_{i \in f} a_i c_i^{-\frac{\alpha}{1-\alpha}} = \sum_{i \in f} a_i c_i^{1-\sigma} = T^f.$$

Therefore, $\Lambda = (T^f)^{\frac{1}{1-\sigma}}$. Recall that $\Lambda = \frac{\sum_{j \in f} c_j q_j}{Q^f}$. This implies that firm f's implied production technology for the composite commodity has constant returns to scale, and that firm f's constant unit cost is equal to $(T^f)^{\frac{1}{1-\sigma}}$. Put differently, firm f's productivity for the composite commodity is $(T^f)^{\frac{1}{\sigma-1}}$.

The indirect utility approach. Firm f is tasked to deliver inclusive value V^f in a profit-maximizing way. That is, firm f solves maximization problem

$$\max_{(p_j)_{j \in f}} \sum_{j \in f} (p_j - c_j) \frac{-h'_j}{\mathrm{e}^{V^f}} \quad \text{s.t.} \quad \log \sum_{j \in f} h_j(p_j) = V^f.$$

It is straightforward to show that firm f's profile of prices must satisfy the constant ι-markup property: There exists μ^f such that $p_j - c_j = \lambda \mu^f$ (resp. $\sigma \frac{p_j - c_j}{p_j} = \mu^f$) in the MNL (resp. CES) case for every $j \in f$.

The optimal value of μ^f is pinned down by the inclusive-value constraint:

$$\log \sum_{j \in f} h_j(r_j(\mu^f)) = V^f.$$

This yields $\mu^f = \log T^f - V^f$ in the MNL case, and $\mu^f = \sigma \left(1 - \left(T^f \mathrm{e}^{-V^f}\right)^{\frac{1}{1-\sigma}}\right)$ in the CES case. Plugging this value of μ^f into the objective function, we find that firm f makes a profit of $\log T^f - V^f$ in the MNL case, and $(\sigma - 1)\left(1 - \left(T^f \mathrm{e}^{-V^f}\right)^{\frac{1}{1-\sigma}}\right)$ in the CES case. In both cases, a firm with a higher T^f delivers inclusive value V^f in a more efficient way.

Next, we study the impact of trade liberalization on domestic industry-level productivity:

Proposition XII. *With CES or MNL demands, a trade liberalization raises the domestic industry-level productivity.*

Proof. Assume without loss of generality that $\mathcal{F} = \{1, \ldots, n\}$, and that $T^1 \leq \ldots \leq T^n$. Let $(s^f)_{1 \leq f \leq n}$ (resp. $(s'^f)_{1 \leq f \leq n}$) be the pre-trade liberalization (resp. post-trade liberalization) vector of market shares. Define also Φ and Φ' as the pre- and post-trade liberalization industry-level productivity, respectively. By Proposition VIII, we have that $s'^f s^g \leq s^f s'^g$ whenever $f \leq g$.

For every $1 \leq f \leq n$, define $w^f = s^f / \sum_{g=1}^n s^g$ and $w'^f = s'^f / \sum_{g=1}^n s'^g$. We interpret $w \equiv (w^f)_{1 \leq f \leq n}$ and $w' \equiv (w'^f)_{1 \leq f \leq n}$ as discrete probability distributions over $\{1, \ldots, n\}$. We claim that w' first-order stochastically dominates w. To see this, let $F \in \{1, \ldots, n\}$, and note that

$$\begin{aligned}\sum_{f=1}^{F} w'^f \le \sum_{f=1}^{F} w^f &\iff \frac{\sum_{f=1}^{F} s'^f}{\sum_{f=1}^{n} s'^f} \le \frac{\sum_{f=1}^{F} s^f}{\sum_{f=1}^{n} s^f}, \\ &\iff \sum_{f=1}^{F} s'^f \sum_{g=F+1}^{n} s^g \le \sum_{f=1}^{F} s^f \sum_{g=F+1}^{n} s'^g, \\ &\iff \sum_{f=1}^{F} \sum_{g=F+1}^{n} \underbrace{\left(s'^f s^g - s^f s'^g\right)}_{\le 0} \le 0,\end{aligned}$$

which holds true by Proposition VIII. This confirms that w' first-order stochastically dominates w.

Since functions $\varphi(\cdot)$ and $f \mapsto T^f$ are increasing, it follows that

$$\Phi' = \sum_{f=1}^{n} w'^f \varphi(T^f) \ge \sum_{f=1}^{n} w^f \varphi(T^f) = \Phi. \qquad \square$$

Welfare

Proposition XIII. *Under monopolistic competition with CES or MNL demands, a trade liberalization raises domestic welfare.*

Proof. Recall that, under monopolistic competition, every firm sets a ι-markup of 1. If demand is CES, then

$$\begin{aligned}W(H^0) &= \log\left(H^0 + \sum_{j\in\mathcal{N}} a_j c_j^{1-\sigma}\left(\frac{\sigma-1}{\sigma}\right)^{1-\sigma}\right) + \sum_{f\in\mathcal{F}} \frac{\sigma-1}{\sigma} \frac{\sum_{k\in f} \sigma \frac{p_k - c_k}{p_k} a_k c_k^{1-\sigma}\left(\frac{\sigma-1}{\sigma}\right)^{1-\sigma}}{H^0 + \sum_{j\in\mathcal{N}} a_j c_j^{1-\sigma}\left(\frac{\sigma-1}{\sigma}\right)^{1-\sigma}}, \\ &= \log\left(H^0 + \alpha^{\frac{1}{1-\alpha}} \sum_{f\in\mathcal{F}} T^f\right) + \alpha \frac{\alpha^{\frac{1}{1-\alpha}} \sum_{f\in\mathcal{F}} T^f}{H^0 + \alpha^{\frac{1}{1-\alpha}} \sum_{f\in\mathcal{F}} T^f},\end{aligned}$$

where $\alpha = (\sigma-1)/\sigma$, and

$$W'(H^0) = \frac{H^0 + (1-\alpha)\alpha^{\frac{1}{1-\alpha}} \sum_{f\in\mathcal{F}} T^f}{\left(H^0 + \alpha^{\frac{1}{1-\alpha}} \sum_{f\in\mathcal{F}} T^f\right)^2} > 0.$$

Under MNL demand,

$$W(H^0) = \log\left(H^0 + \sum_{j\in\mathcal{N}} \mathrm{e}^{-1}\, \mathrm{e}^{\frac{a_j - c_j}{\lambda}}\right) + \sum_{f\in\mathcal{F}} \frac{\sum_{k\in f} \frac{p_k - c_k}{\lambda}\, \mathrm{e}^{\frac{a_k - c_k}{\lambda}}\, \mathrm{e}^{-1}}{H^0 + \sum_{j\in\mathcal{N}} \mathrm{e}^{-1}\, \mathrm{e}^{\frac{a_j - c_j}{\lambda}}},$$

$$= \log\left(H^0 + \mathrm{e}^{-1}\sum_{f\in\mathcal{F}} T^f\right) + \frac{\sum_{f\in\mathcal{F}} \mathrm{e}^{-1}\, T^f}{H^0 + \mathrm{e}^{-1}\sum_{f\in\mathcal{F}} T^f}.$$

Therefore,

$$W'(H^0) = \frac{H^0}{\left(H^0 + \mathrm{e}^{-1}\sum_{f\in\mathcal{F}} T^f\right)^2} > 0. \qquad \square$$

XI Table of Symbols and Notations

Market-level notations	
H	Aggregator, sufficient statistic for consumer surplus
$\Gamma(H)$	Aggregate fitting-in function
$\Omega(H)$	$\Gamma(H)/H$
$\mathcal{N}$	Set of products
$\mathcal{F}$	Set of firms
Firm-level notations	
μ^f	Firm f's ι-markup
$m^f(H)$	Firm f's fitting-in function
$\bar{\mu}^f$	$\max k \in f \bar{\mu}_k$ The highest ι-markup that firm f can sustain
ω^f	$(\mu^f - 1)/\mu^f$
T^f	Firm f's type (CES / MNL demands)
Product-level notations	
$\mathcal{H}$	The set of $\mathcal{C}^3$, strictly decreasing and log-convex functions from $\mathbb{R}_{++}$ to $\mathbb{R}_{++}$
$\mathcal{H}^\iota$	The set of functions in $\mathcal{H}$ that satisfy Assumption 1
h_k	Exponential of indirect subutility derived from product k
$-h'_k/h_k$	Conditional demand for product k
$h_k/\sum_{j\in\mathcal{N}} h_j$	Choice probability for product k
ι_k	$p_k h''_k(p_k)/(-h'_k(p_k))$, elasticity of monopolistic competition demand
$\bar{\mu}_k$	$\lim_\infty \iota_k$, the highest ι-markup that product k can sustain
γ_k	h'^2_k/h''_k
ρ_k	h_k/γ_k
θ_k	h'_k/γ'_k
χ_k	$(\iota_k - 1)/(\iota_k)$
$\nu_k(p_k)$	$\iota_k(p_k)(p_k - c_k)/p_k$, ι-markup on product k
$r_k(\mu^f)$	$\nu_k^{-1}(\mu^f)$, pricing function
p_k^{mc}	$r_k(1)$, product k's price under monopolistic competition
$\underline{p}_k$	$\inf\{p_k > 0 : \ \iota_k(p_k) > 1\}$

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